APPLICATIONS OF COVARIANT SCATTERING THEORY

Dissertation

for the degree of

DOCTOR OF PHILOSOPHY

submitted by

DIKRAN HAGOP TCHRAKIAN, B.Sc. (Lond.), A.R.C.S.

Tait Institute of Mathematical Physics,

University of Edinburgh,

September, 1967.

CONTENTS

CONTENTS (Contd.)

Page

.....It is often said that spin is only
an inessential complication. Nevertheless,
it appears that except in simple cases,
a certain amount of complication is, if
not essential, at least unavoidable......

from A.O. Barut, I. Muzinich, D.N. Williams
Physical Review, Volume 130, Number 1,
page 443.

ACKNOWLEDGEMENTS

I am deeply indebted to Dr. D.J. Candlin and Professor A.O. Barut, whose guidance and help has played an important role in the development of this dissertation. I would like to thank Professor N. Kemmer, F.R.S., for his hospitality at the Tait Institute. My thanks are also due to Mr. P. Bradfield for his assistance in the numerical computations, and those members of the Tait Institute with whom I have had many conversations on subjects in this dissertation.

I acknowledge gratefully a Scholarship from the C. Gulbenkian Foundation during my first two years, and a Studentship from the University of Edinburgh during the last two terms of my third year at the Tait Institute.

CHAPTER I

NOTES ON THE LORENTZ GROUP

1. Lorentz Transformations

The transformations forming the Inhomogeneous Lorentz Group (I.L.G.) consist of the group of homogeneous Lorentz transformations (real Lorentz transformations of space-time and 3-dimensional rotations) followed by real space-time translations,

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu} + a^{\mu} \; ; \; (\nu,\mu = 0,1,2,3) \tag{1}$$

such that the $\Lambda^{\mu}{}_{\nu}$ operations leave

$$x^2 = x_o{}^2 - x_1{}^2 - x_2{}^2 - x_3{}^2 \tag{2}$$

invariant. This conditions restricts $^{\mu}{}_{\nu}$ to satisfy the condition

$$\Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma}\, g_{\mu\nu} = g_{\rho\sigma} \tag{3}$$

where $g_{\mu\nu}$ is the metric tensor

$$\begin{aligned} g_{oo} &= -g_{11} = -g_{22} = -g_{33} = 1 \\ g_{\mu\nu} &= 0 \quad \text{for} \quad \mu \neq \nu \; . \end{aligned} \tag{4}$$

The invariant form (2) is a result of a contraction between upper and lower indices, so that $g_{\mu\nu}$ and $g^{\mu\nu}$ lower and raise the indices to give us (2).

$$x^2 = g_{\mu\nu}\, x^{\mu} x^{\nu} = g^{\mu\nu}\, x_{\mu} x_{\nu} \; . \tag{5}$$

Using the relation

$$g^{\mu\lambda}\, g_{\lambda\nu} = \delta^{\mu}_{\ \nu} , \qquad (6)$$

we can write (3) in the form

$$\Lambda^{\mu}{}_{\rho}\, \Lambda_{\mu}{}^{\sigma} = \delta_{\rho}{}^{\sigma} \qquad (7)$$

from which it follows that the inverse of $\quad ^{\mu}{}_{\nu}$ is given by

$$[\Lambda^{-1}]^{\mu}{}_{\nu} = \Lambda_{\nu}{}^{\mu} \qquad (8)$$

The condition (3) is analogous to the condition of orthogonality for the real rotation group, and can be written in the form

$$\Lambda^{T}\ G\ \Lambda = G \qquad (9)$$

and taking the determinant of this, we have

$$\det \Lambda = \pm 1 . \qquad (10)$$

In the following, we shall be interested in what are called the proper Lorentz transformations. It is possible to connect any two proper transformations by going through a continuous path, and in this case the determinant in (10) is equal to +1. We neglect the case with -1 in (10) because this corresponds to a discontinuous transformation, e.g. a reflection given by with determinant -1. These transformations are called improper.

It also follows from (10) that

$$|\Lambda^{0}{}_{0}| \geqslant 1 ,$$

the two cases being called orthochronous and non orthochronous respectively. As in the following, we shall be interested only in particles of $m^2 \quad 0$, our attention will be restricted to the positive "light cone", where the time component cannot change sign under a Lorentz transformation. Thus we consider only the following Lorentz transformations

$$\text{proper} \quad : \quad \det\left(\Lambda^{\mu}{}_{\nu}\right) = +1$$

$$\text{orthochronous} \quad : \quad \Lambda^{0}{}_{0} > 0 \; .$$

This completes our definition of the Lorentz transformations.

Finally we demonstrate that inhomogeneous Lorentz transformations form a group, simply by performing two consecutive transformations $\{\Lambda_1, a_1\}$ and $\{\Lambda_2, a_2\}$, which give

$$\{\Lambda_2, a_2\}\{\Lambda_1, a_1\} = \{\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2\} \tag{11}$$

which is itself an inhomogeneous Lorentz transformation as required by the group law.

The inverse follows then to be

$$\{\Lambda, a\}^{-1} = \{\Lambda^{-1}, -\Lambda^{-1}a\}$$

and the identity element is $\{1, 0\}$.

2. The Lie Algebra of I.L.G.

The element, of the Unitary representation of I.L.G., corresponding to an infinitesimal element $1 + \omega, \varepsilon$ of the group, can be written to first order in the infinitesimal group parameters as

$$U\left[1 + \omega, \varepsilon\right] = 1 + \tfrac{i}{2}\,\omega^{\mu\nu} J_{\mu\nu} - i\,\varepsilon^{\mu} P_{\mu} \tag{12}$$

As a consequence of (3)

$$\omega_{\mu\nu} = -\,\omega_{\nu\mu}$$

therefore $\omega_{\mu\nu}$ constitute six real parameters (three Lorentz transformations ω_{io}, three rotations ω_{ij} $i \neq j \neq 0$) which together with four from ε^{μ} give us ten real parameters.

As these are unitary representations, it follows that the generators must be hermitian,

$$J_{\mu\nu} = J^{+}_{\mu\nu}, \quad P_{\mu} = P^{+}_{\mu} \tag{14}$$

and that

$$J_{\mu\nu} = -\,J_{\nu\mu} \quad , \tag{15}$$

thus the I.L.G. has ten infinitesimal generators.

To derive the algebra these infinitesimal operators generate, we must surely invoke the group property (11). To this end we apply an arbitrary transformation $U\left[\Lambda\, , a\right]$ to both sides of the operator equation (12), and use the group property (11). For the left-hand side of (12) we get

$$U[\Lambda,a]\,U[1+\omega,\varepsilon]\,U^{-1}[\Lambda,a] = 1 + \tfrac{i}{2}(\Lambda\omega\Lambda^{-1})^{\mu\nu} J_{\mu\nu} - i\left(-\Lambda\omega\Lambda^{-1}a + \Lambda\varepsilon\right)^{\mu} P_{\mu} \tag{16}$$

and for the right-hand sides

$$\begin{aligned} U[\Lambda,a]\left[1+\tfrac{i}{2}\omega^{\mu\nu}J_{\mu\nu} - i\varepsilon^{\mu}P_{\mu}\right]U^{-1}[\Lambda,a] \\ = 1 + \tfrac{i}{2}\omega^{\mu\nu}\,U[\Lambda,a]\,J_{\mu\nu}\,U^{-1}[\Lambda,a] \\ - i\,\varepsilon^{\mu}\,U[\Lambda,a]\,P_{\mu}\,U^{-1}[\Lambda,a]. \end{aligned} \tag{17}$$

Comparing coefficients of $\omega^{\mu\nu}$ and ε^{μ} in (16) and (17) we get,

$$U[\Lambda,a]\,J_{\mu\nu}U^{-1}[\Lambda,a] = \Lambda^{\rho}{}_{\mu}\,\Lambda^{\sigma}{}_{\nu}\left(J_{\rho\sigma} + a_{\sigma}J_{\rho} - a_{\rho}J_{\sigma}\right) \tag{18}$$

$$U[\Lambda,a]\,P_{\mu}\,U^{-1}[\Lambda,a] = \Lambda^{\rho}{}_{\mu}\,P_{\rho} \tag{19}$$

It is clear from these equations that $J_{\mu\nu}$ transforms as a tensor and P_{μ} as a vector under homogeneous Lorentz transformations, and that further P_{μ} is translation invariant.

We now let $\{\Lambda, a\} = \{1 + \lambda, \xi\}$, with λ and ξ infinitesimal parameters, and by equating the coefficients of λ in (18), and (19), and the coefficients of ξ in (19), we get from each respectively,

$$i\left[J_{\alpha\beta}, J_{\mu\nu}\right] = g_{\beta\mu}J_{\alpha\nu} - g_{\alpha\mu}J_{\beta\nu} - g_{\beta\nu}J_{\alpha\beta} + g_{\alpha\nu}J_{\beta\mu} \tag{20}$$

$$i[P_\mu , J_{\alpha\beta}] = g_{\alpha\mu} P_\beta - g_{\beta\mu} P_\alpha \tag{21}$$

$$[P_\alpha , P_\mu] = 0 \tag{22}$$

(20), (21) and (22) form the Lie Algebra of I.L.G.

3. Definition of Spin Operators.

We have seen above that P_μ's commute with each other. We now define another operator, to complete the list of commuting operators,

$$W_\mu = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} J^{\nu\rho} P^\sigma \tag{23}$$

which has the following commutation relation (C.R)

$$[W_\mu , P_\nu] = 0 \tag{24}$$

$$[W_\mu , W_\nu] = -i\varepsilon_{\mu\nu\rho\sigma} W^\rho P^\sigma \tag{25}$$

$$[J_{\mu\nu} , W_\lambda] = i(g_{\mu\nu} W_\nu - g_{\nu\lambda} W_\mu) \tag{26}$$

and $$W_\mu P^\mu = 0 \tag{27}$$

$\varepsilon_{\mu\nu\rho\sigma}$ - the completely antisymmetric tensor.

It follows from these C.R. that $P.P = P^2$ and $W.W = W^2$ commute with all the generators of I.L.G. It can be shown that these two exhaust the list of all functions which commute with all the generators.

We take P to be diagonal with eigenvalue p, such that $p^2 = m^2$ (m^2 fixed). From this point, we restrict the discussion to $p^2 > 0$, because the only cases we shall be interested in are $p^2 \geqslant 0$, and the $p^2 = 0$ case is sufficiently closely connected with the $p^2 > 0$ case, so we shall leave its details till paragraph 7.

Now $W(p).p = 0$ means that it is possible to expand $W(p)$ along a basis of independent vectors orthogonal to p. To this end we introduce the vectors n_i^{μ} ($i = 1,2,3$ - spacelike) which are orthogonal to p^{μ}

$$p_{\mu}\, n_i^{\mu} = 0 \tag{28}$$

and $$n_i^{\mu}\, n_{j\mu} = -\delta_{ij} \tag{29}$$

so that $W(p)$ is expanded in the form

$$W^{\mu}(p) = \sum_{i=1}^{3} W_i(p)\, n_i^{\mu} \tag{30}$$

$$W_i(p) = -W_{\mu}(p)\, n_i^{\mu} \tag{31}$$

Using (31) and the C.R. (25) we get

$$\left[S_i(p),\ S_j(p) \right] = i\, \varepsilon_{ijk}\, S_k(p) \tag{32}$$

where $$S_i(p) = \frac{W_i(p)}{m} \tag{33}$$

which is the well known SU(2,C) or O(3) algebra, with

$$ {}_{i}\quad s_i^{2}(p) \;=\; -\,\frac{W^2}{m^2} \;=\; s(s+1)\;\; ; \quad s = 0,\ \tfrac{1}{2},\ 1\ldots\ldots \tag{34}$$

S is another diagonalisable operator, whose representations $(S_1\ S_3)$, with eigenvalue s_3, label the single particle states in Hilbert space, along with the four-momenta p_μ.

For massless particles, $W^2 = 0$ gives us, together with $W(p).p = 0$ and $p^2 = 0$, the solution

$$ W(p) \;=\; \lambda(p)\ p \tag{35}$$

where $\lambda(p)$ is a scalar. This is consistent with the fact that the spin projection of a massless particle is fixed for all Lorentz frames.

4. Vectors, Rays and the Covering Group

In this paragraph, we shall just quote some very important results, which we shall use without proof, subsequently.

These are, the representation of physical states in quantum mechanics, and consequences of some topological properties of the parameter space of the group of Lorentz transformations.

We shall assume[1] that there exists a Hilbert space $\mathcal{H}$, in which vectors $|\Phi\rangle$ have a normalised unitary scalar product. $|\Phi\rangle$ is defined in terms of the vectors $|p, \sigma\rangle$, labelled by the continuous momentum, and discrete spin quantum numbers (cf. paragraph 3),

(36)

where $\phi(p, \sigma) = \langle p, \sigma | \Phi \rangle$ is the wavefunction, with the following "normalisation",

$$\langle p, \sigma | p', \sigma' \rangle = \delta^{(3)}(\vec{p} - \vec{p}')\ \delta_{\sigma\sigma'} . \tag{37}$$

It follows from (36) and (37) that the scalar product in is

$$\langle \Phi | \Psi \rangle = \sum_{\sigma} \int \frac{d^3\vec{p}}{p_0}\, \varphi^*(p,\sigma)\, \psi(p,\sigma) \tag{38}$$

where $$\int \frac{d^3\vec{p}}{p_0} f(\vec{p}) = \int d^4p\ \theta(p_0)\, \delta(p^2 - m^2)\, f(p_0, \vec{p})$$

defines the Lorentz Invariant measure.

Having defined $\mathcal{H}$ with vectors $|\Phi\rangle$ possessing a normalised unitary scalar product, we shall, in the following, deal entirely with the vectors $|p, \sigma\rangle$, keeping in mind that the proper normalised scalar product is given by (38).

Now in quantum mechanics, the measurable quantities are the moduli squares of the scal products in $\mathcal{H}$, which are the transition probabilities between two states (vectors), so that, all vectors differing from each other only by a phase factor $e^{i\alpha}$ $(0 < \alpha \leqslant 2\pi)$, will give rise to the same measurement. Such collections of vectors are called rays. Thus the state of a system is not described by a vector $|\Phi\rangle$ but by a ray $|\underline{\Phi}\rangle = e^{i\alpha}|\Phi\rangle$.

The statement of a symmetry operation Γ, is

$$|\langle \underline{\Phi} | \underline{\Psi} \rangle|^2 = |\langle {}^{\Gamma}\underline{\Phi} | {}^{\Gamma}\underline{\Psi} \rangle|^2 \tag{39}$$

which is the conservation of the transition probability $|\langle\Gamma\Phi\rangle$ being the ray after operation of Γ .

We next state Wigner's Theorem( ):

To every symmetry operation Γ on the rays, there corresponds a unitary or antiunitary additive operator U, which acts on the vectors in $\mathcal{H}$. U^+ is defined by $\langle\Phi|U^\dagger|\Psi\rangle = \langle U\Phi|\Psi\rangle$ in the unitary case(*) and $\langle\Phi|U^\dagger|\Psi\rangle = \langle U\Phi|\Psi\rangle^*$ in the antiunitary case.

If now we suppose that the symmetry operations form a group G , then there exists a unitary operator U(e) corresponding to the identity element e. Now the U(g) being defined uniquely up to a phase factor, we have

$$U(e) = \omega \mathbb{1} \tag{39}$$

ω being the phase factor, which we shall choose $\omega = 1$. Further, it can be shown, by using the additivity property of U(g) that

$$U(g_1)\, U(g_2) = \omega(g_1, g_2)\, U(g_1, g_2) \;\ldots\ldots \tag{40}$$

where g_1 and g_2 are elements of the group G .

We now turn out attention to I.L.G., which consists of the translations and Lorentz transformations. The latter form the group O(3, 1) which is topologically a four-dimensional "rotation" group, with metric $g_{\mu\nu}$ defined by (4).

If $\mathcal{U}(a, \Lambda)$ be the unitary operators representing the elements of I.L.G., i.e.

---

* In the following we shall be using only Unitary operators.

$$\mathcal{U}(a,1)\,\mathcal{U}(0,\Lambda) \equiv \mathcal{U}(a,\Lambda)$$

then by (40)

$$\begin{aligned}
\mathcal{U}(a,1)\,\mathcal{U}(b,1) &= \omega(a,b)\,\mathcal{U}(a+b,1) \\
\mathcal{U}(0,\Lambda)\,\mathcal{U}(a,1) &= \omega(\Lambda,a)\,\mathcal{U}(\Lambda a,1)\,\mathcal{U}(0,\Lambda) \qquad (41)\\
\mathcal{U}(0,\Lambda_1)\,\mathcal{U}(0,\Lambda_2) &= \omega(\Lambda_1,\Lambda_2)\,\mathcal{U}(0,\Lambda_1\Lambda_2)
\end{aligned}$$

It has been shown by Wigner( ) that ω(a,b) and ω(,a) can be assumed to be 1, and that $\omega(\ _1, \ _2) = \pm 1$, by examination of the topological properties of the parameter space of O(3,1).

Finally, we remark that ± (0,) form a <u>single valued</u> representation of SL(2,C), the group of complex unimodular two-dimentional matrices, A. This group is also called the universal covering group of the homogeneous Lorentz transformations. We shall denote the unitary operators representing the elements of I.L.G. in the two-dimensional complex space U(a, A), and rewrite (41) in this space as

$$\begin{aligned}
U(a, 1)\ U(b, 1) &= U\ a + b, 1 \\
U\ 0, A\quad U\ a, 1 &= U\quad a, 1\quad U\ 0, A \qquad (42)\\
U\ 0, A_1\quad U\ 0, A_2 &= U\ 0, A_1A_2 \qquad .
\end{aligned}$$

It is in fact the ± sign in the third member of (41) that gives rise to half-integral spins. The situation is the same in this respect, also for $O(3)$, the group of pure rotations.

In the following, unless otherwise stated, we shall be using exclusively those representations of the Lorentz group that are in $SL(2, C)$. These matrices are given by eight real (four complex) parameters with two real (one complex) relation(s) of unimodularity between them, thus, ending up with six real parameters corresponding to $\omega_{\mu\nu}$ $(= -\omega_{\nu\mu})$ cf. (13).

A Lorentz transformation $A[\Lambda]$ is given by

$$A\ \sigma^\mu x_\mu\ A^\dagger = \sigma^\mu x'_\mu = \sigma^\mu (\Lambda_\mu{}^\nu x_\nu) \tag{43}$$

where the one-to-two so called homomorphism between the elements of the Lorentz group, $\Lambda^\mu{}_\nu$, and the elements of the 2×2-unimodular group, A, is given by

$$\Lambda^\mu{}_\nu [\pm A] = \tfrac{1}{2} \mathrm{Tr}\, (\tilde{\sigma}^\mu A\, \sigma_\nu\, A^\dagger) \tag{44}$$

where $\sigma_0 = 1$, $\sigma_i \equiv$ the Pauli spin-matrices, and $\sigma = (\sigma_0, -\vec{\sigma})$.

5. Unitary Representations of I.L.G.

According to the last paragraph, we can examine the effect of a symmetry operation on the rays by considering the operation of a Unitary Operator onto the vectors $|p, \sigma\rangle$.

Firstly, because the translations from an Abelian subgroup of I.L.G., we can represent them by a one-dimensional representation,

$$U[a,1]\,|p,\sigma\rangle = e^{ip.a}\,|p,\sigma\rangle \qquad (45)$$

We now introduce an operator P A , such that

$$P[A]\,|p,\sigma\rangle = |\Lambda p,\sigma\rangle \qquad (46)$$

where $A \equiv A[\Lambda]$ is an element of SL(2, C). It follows from this definition that

$$\begin{aligned} U[a,1]\;P[A^{-1}]\,|p,\sigma\rangle &= U[a,1]\,|\Lambda^{-1}p,\sigma\rangle \\ &= e^{i\Lambda^{-1}p.a}\,|\Lambda^{-1}p,\sigma\rangle \\ P[A^{-1}]\,U[\Lambda a,1]\,|p,\sigma\rangle &= e^{i\Lambda a.p}\,P[A^{-1}]\,|p,\sigma\rangle \\ &= e^{i\Lambda a.p}\,|\Lambda^{-1}p,\sigma\rangle \end{aligned} \qquad (47)$$

but it is easily seen from (8) that $\Lambda^{-1}p.a = \Lambda a.p$ so that,

$$U[a,1]\,P[A^{-1}] = P[A^{-1}]\,U[\Lambda a,1] \qquad (48)$$

From the second member of (42) and (48), it follows that

$$\left[U[\Lambda a,1]\,,\,Q[A]\right] = 0 \qquad (49)$$

where

$$Q[A] = U[0,A]\;P[A^{-1}]. \qquad (50)$$

It follows from (49) that $Q[A]$ commutes with all $U[a,1]$, and since the exponentials (cf. (45)) form a complete set of functions of p_μ, therefore $Q[A]$ commutes with all other functions of p as well. This means that $Q[A]$, although it is a function of p, it is also only a constant matrix in the space of p, and is an operator in the space of σ alone. We express this by

$$Q[A]\,|p,\sigma\rangle = \sum_{\sigma'} Q[p,A]_{\sigma\sigma'}\,|p,\sigma'\rangle . \tag{51}$$

From (48) and (49) we get

$$U[0,A]\,|p,\sigma\rangle = \sum_{\sigma'} Q[p,A]_{\sigma\sigma'}\,|\Lambda p,\sigma'\rangle \tag{52}$$

To determine $Q[P, A]_{\sigma\sigma'}$, we examine its group properties by using the group operations of the homogeneous Lorentz transformations (c.f. 3rd member of (42)) in (52). This yields

$$\sum_{\sigma''} Q[p,A_1]_{\sigma\sigma''}\,Q[\Lambda_1 p, A_2]_{\sigma''\sigma'} = Q[p, A_1 A_2]_{\sigma\sigma'} \tag{53}$$

from which it is clear that

$$Q[p,\mathbb{1}]_{\sigma\sigma'} = \delta_{\sigma\sigma'} .$$

Obviously, $Q[p, A]$ do not form a group for $A \in SL(2C)^{\uparrow}_{+}$, but if we consider the subgroup of Lorentz transformations including all $\mathring{A}$, such that

$$\Lambda[\mathring{A}_1]\,\mathring{p} = \mathring{p} \quad ; \quad \Lambda[\mathring{A}_2]\,\mathring{p} = \mathring{p} \tag{54}$$

where $\mathring{p}$ is some standard momentum four-vector, then

$$\sum_{\sigma''} Q[\mathring{p}, \mathring{A}_1]_{\sigma\sigma''} Q[\mathring{p}, \mathring{A}_2]_{\sigma''\sigma'} = Q[\mathring{p}, \mathring{A}_1 \mathring{A}_2]$$

or $$q[\mathring{A}_1]\, q[\mathring{A}_2] = q[\mathring{A}_1 \mathring{A}_2] \tag{55}$$

Thus the subgroup of all matrices $\mathring{A}$ have representations satisfying the group property (55). It is called the "little group".

We now proceed to prove that if the representation $q[\mathring{A}]$ of the "little group" determines the representation of the whole group. It must however at this stage be made clear if the system under consideration has got timelike ($p^2 > 0$), lightlike ($p^2 = 0$), spacelike ($p^2 < 0$) or null ($p_\mu = 0$) four-momentum square. This is because we are interested in representations, the states of which vanish except for such momenta as can be obtained from each other by homogeneous Lorentz transformations Therefore the class of p^2 must be specified in addition to the representation of the "little group".

To prove this, we introduce the "boost" operators defined by

$$U\left[0, B(\vec{p})\right] \left|\mathring{p}, \sigma\right\rangle = \left|p, \sigma\right\rangle \tag{56}$$

where $B(\vec{p})$ is *an* SL(2C) element taking $\mathring{p}$ to p, while in the O(3, 1) space, we shall denote these elements by $L(p)^\mu{}_\nu$.

(54) is equivalent to having set

$$Q\left[p, B^{-1}(p)\right] = 1 \tag{57}$$

We now write

$$U[0,A] = U[0,B(\vec{\Lambda p})]U[0,B^{-1}(\vec{\Lambda p})]U[0,A]U[0,B(\vec{p})]U[0,B^{-1}(\vec{p})]$$
$$= U[0,B(\vec{\Lambda p})]\, q[\mathring{A}]\, U[0,B^{-1}(\vec{p})] \tag{58}$$

whence

$$U[0,A]|p,\sigma\rangle = U[0,B(\vec{\Lambda p})]\, q[\mathring{A}]\, U[0,B^{-1}(\vec{p})]|p,\sigma\rangle \tag{59}$$

where $q[\mathring{A}]$ being a representation of the little group, operates only in the σ-space, thus

$$U[0,A]|p,\sigma\rangle = \sum_{\sigma'} q[\mathring{A}]_{\sigma\sigma'}|\Lambda p,\sigma'\rangle \tag{60}$$

and

$$U[a,A]|p,\sigma\rangle = e^{ip.a}\sum_{\sigma'} q[\mathring{A}]_{\sigma\sigma'}|\Lambda p,\sigma'\rangle. \tag{61}$$

Finally we remark that it is easy to see by use of (56) and (59) that the representations of the little groups leaving $\mathring{p}$ and $\mathring{p}'$ invariant respectively, are identical.

Formula (61) gives us the transformation of states under the group of inhomogeneous Lorentz transformations.

6. The Representation of the Little Groups

In this paragraph we shall classify the representations of the little groups for the cases $p^2 > 0$ and $p^2 = 0$.

(i) $p^2 = m^2 > 0$:

To find that subgroup of the group of all homogeneous Lorentz transformations which leaves $\mathring{p}$ invariant, we require that the following equation be satisfied for any element $\mathring{A}$ of that subgroup:

$$\mathring{A}\, \sigma_\mu\, \mathring{p}^\mu\, \mathring{A}^+ \quad = \quad \sigma_\mu\, \mathring{p}^\mu \tag{62}$$

which is just a special case of (43).

As the choice of $\mathring{p}$ is arbitrary, we choose it in this case to be the rest frame of the system (single particle) concerned: $\mathring{p} \;=\; (m,\, 0,\, 0,\, 0)$, whence (62) reduces to

$$A\, A^+ \quad = \quad 1 \tag{63}$$

This means that the elements of the little group must be unitary, that is, they belong to the SU(2, C) subgroup of SL(2, C). Thus, the group of invariance in the rest frame of the particle is the Rotation Group, and hence the states are labelled by spin (intrinsic) angular momentum quantum numbers. This can be a definition of spin, in complete accordance with the content of paragraph 3. The effect of a Lorentz transformation, on a state will be

$$U[0,A]\,|p,\sigma\rangle = \sum_{\sigma'} D_{\sigma'\sigma}[\mathring{A}]\,|\Lambda p,\sigma'\rangle\,. \tag{64}$$

We shall now quote some useful results from the theory of representations of SU(2, C), which is the covering group of the Rotation Group, O(3).

The matrices representing the elements of the "little Group", $D^{(j)}[\mathring{A}]$, where j determines the dimensionality of the angular momentum representation, $(2j + 1)$ components of spin, are unitary, and therefore, from the following different representations

$$D^{(j)} \quad D^{(j)*} \quad D^{(j)T-1} \quad D^{\dagger -1} \tag{65}$$

the first and the fourth, as well as the second and the third are the same. Furthermore, the two different representations, say the first and second, are related by a similarity transformation

$$D^{(j)}[\mathring{A}]^{*} = C\, D^{(j)}[\mathring{A}]\, C^{-1} \tag{66}$$

where C is a $(2j+1)\times(2j+1)$ unitary matrix with

$$C\, C^{*} = (-)^{2j} \qquad , C\, C^{\dagger} = 1 \tag{67}$$

The matrix C is the metric spinor, that is, it defines a scalar product in spinor space. Had we considered the spinors the set (65) act on

$$\xi_{\alpha} \quad \xi_{\dot{\alpha}} \quad \xi^{\alpha} \quad \xi^{\dot{\alpha}} \tag{68}$$

we would have seen that for unitary representations an upper dotted (undotted) index is equivalent to a lower undotted (dotted) index, from which it follows immediately that C and C^{-1} are raising and lowering (metric) operators respectively. With the usual phase conventions, C can be taken as the matrix

$$C^{(j)}_{\sigma\sigma'} = (-)^{j+\sigma} \delta_{\sigma', -\sigma} . \tag{69}$$

Lastly we give the Lie Algebra for SU(2, C). The rotations form a three-parameter group, and hence are generated by three infinitesimal generators $J_i \equiv \vec{J}$ (i = 1,2,3 spacelike), which have the following well known Lie Algebra

$$[J_i, J_j] = i\, \varepsilon_{ijk} J_k \tag{70}$$

where ε_{ijk} is the totally antisymmetric tensor. The (2j + 1) dimensional representations are then given by

$$\begin{aligned} \langle\sigma'| J_1^{(j)} \pm i J_2^{(j)} |\sigma\rangle &= \delta_{\sigma', \sigma\pm 1} [(j \mp \sigma)(j \pm \sigma + 1)]^{1/2} \\ \langle\sigma'| \quad J_3^{(j)} \quad |\sigma\rangle &= \sigma\, \delta_{\sigma\sigma'} \end{aligned} \tag{71}$$

(ii) $p^2 = m^2 = 0$:

Choosing $\mathring{p} = (\omega, 0, 0, \omega)$ as our standard four-vector, we proceed as in (i), by substituting this $\mathring{p}$ into (62). It then follows that an element of the little group would be of the following form

$$A = \begin{bmatrix} e^{i\varphi/2} & z\, e^{-i\varphi/2} \\ 0 & e^{-i\varphi/2} \end{bmatrix} \tag{72}$$

where z is an arbitrary complex number and 0 .

By applying two consecutive such transformations, we find the resultant

$$\mathring{A}(\varphi_1, z_1)\,\mathring{A}(\varphi_2, z_2) = \begin{bmatrix} e^{i(\varphi_1+\varphi_2)/2} & z_2 e^{i(\varphi_1-\varphi_2)/2} + z_1 e^{-i(\varphi_1-\varphi_2)/2} \\ 0 & e^{-i(\varphi_1+\varphi_2)/2} \end{bmatrix} \tag{73}$$

which is obviously another element of this little group. Furthermore, it is clear from the diagonal elements of this matrix that some of the operations of this group are the rotations through ϕ around an axis in a two-dimensional abstract space.

To investigate the structure of this group further, it will be convenient to consider its operations in space-time, although, as we have said before, we shall finally be concerned only with the representations of the subgroup of SL(2, C) in the two-dimensional complex space.

The homogeneous Lorentz transformations $\mathcal{R}^\mu{}_\nu$ constituting the elements of the "little group" must satisfy

$$\begin{aligned} \mathcal{R}^\mu{}_\nu\, \mathring{p}^\nu &= \mathring{p}^\mu \\ \mathring{p} &= (\omega, 0, 0, \omega) \end{aligned} \tag{74}$$

and if we represent these transformations in terms of their infinitesimal versions

$$\mathcal{R}^\mu{}_\nu = \delta^\mu{}_\nu + \Omega^\mu{}_\nu \tag{75}$$

then the following two conditions on the infinitesimal part hold

$$\begin{aligned} \Omega^\mu{}_\nu\, \mathring{p}^\nu &= 0 \\ \Omega^{\mu\nu} &= -\Omega^{\nu\mu}. \end{aligned} \tag{76}$$

(77)

It follows from (76) and (77) that the "little group" is a three parameter group, the three parameters corresponding to the following non-vanishing $\Omega^{\mu\nu}$:

$$\Omega^{12} = -\Omega^{21} = \varphi \tag{78}$$

$$\Omega^{10} = -\Omega^{01} = \Omega^{13} = -\Omega^{31} = \xi_1 \tag{79}$$

$$\Omega^{20} = -\Omega^{02} = \Omega^{23} = -\Omega^{32} = \xi_2 \tag{80}$$

It is clear that (78) is a rotation in a plane perpendicular to the third axis (c.f. (72)) and its generator for the unitary representation is J_3. Calling the other two generators corresponding to the parameters ξ_1 and ξ_2, L_1 and L_2 respectively, and by repeating the procedure in paragraph (2), we find for the Lie Algebra of this little group

$$[J_3, L_1] = iL_2 \tag{81}$$

$$[J_3, L_2] = -iL_1 \tag{82}$$

$$[L_1, L_2] = 0 \tag{83}$$

where

$$U[\mathcal{R}] = 1 + i\varphi J_3 + i\xi_1 L_1 + i\xi_2 L_2 \tag{84}$$

From (83), we see that this group has an Abelian subgroup, which means that its unitary representations will be infinite dimensional. However, we must have finite dimensional representations, because these matrices operate in the spin space (cf. (i)), and the number of spin-projections must be finite. We must thus represent the Abelian "translations", L_1 and L_2, by

$$L_1 \,|\, p, \sigma \rangle \;=\; L_2 \,|\, p, \sigma \rangle \;=\; 0 \tag{85}$$

and thus we end up with the representations

$$U[R] = e^{i J_3 \varphi} \tag{86}$$

The effect of a Lorentz transformation, Λ , on a helicity state defined by

$$J_3 \,|\, p, \lambda \rangle \;=\; \lambda \,|\, p, \lambda \rangle \tag{87}$$

will be

$$U[R(\Lambda)] \,|p, \lambda\rangle = e^{i \lambda \varphi(\Lambda)} \,|\Lambda p, \lambda\rangle \tag{88}$$

which is in agreement with what we sould have got from (72) had we set $\mathbf{z} = 0$.

We must make a remark on (87). Here λ, the helicity, replaces σ, the third component of spin as the eigenvalue labelling a state. The difference is, that λ is the projection of spin along the direction of the momentum. The difference the little groups suffer in this case is due to the redefinition of the "boost" in equation (56) by

$$U[B(\vec{p})]\, U[R(\hat{p})] \;\; |\, \mathring{p}, \lambda \rangle \;=\; |\, p, \lambda \rangle \tag{89}$$

where $R(\hat{p})$ is the rotation taking $\hat{z}$ into $\hat{p}$. The helicity "boosts" for $m^2 > 0$ tend to the helicity "boosts" for $m^2 = 0$ in the limit of $m^2 \longrightarrow 0$. We shall return to these points in greater detail in Section IV.

From (88) we deduce that a particle which travels with the speed of light, i.e. is massless, has the same helicity in all Lorentz frames. We also note that these representations are one-dimensional, and correspond to a particle of spin $j = |\lambda|$. The $(\frac{1}{2}, \frac{1}{2})$-element in (72) clearly refers to a spin $\frac{1}{2}$ particle, and the representation for a spin-j particle is obtained by making use of

$$D^{(j)}[cQ] = c^{2j} D^{(j)}[Q]$$

which follows from (90)

$$D^{(j)} = [j-\tfrac{1}{2} \ \ \tfrac{1}{2} \ \ j] \otimes [j-\tfrac{1}{2} \ \ \tfrac{1}{2} \ \ j] \ D^{(j-\frac{1}{2})} \otimes D^{(\frac{1}{2})} \tag{91}$$

where $[j \ j' \ j'']^{\alpha\beta}_{\gamma}$ is the C.G. coefficient of SU(2,C).

7. Irreducible Representations of H.L.G.

The above representations give us only the transformation properties of spin-states due to a change of Lorentz frame. They are essentially representations of the little groups. In paragraph 2, however, we did have occasion to refer to the actual transformation properties of certain objects under the group of homogeneous Lorentz transformations (H.L.G.). In this paragraph, we shall classify the irreducible representations of this group.

Its Lie Algebra is given in paragraph 2, equation (20). Since the dependent $J_{\mu\nu}$ are six in number, we make the following rearrangement, such that we can deal with this antisymmetric tensor as a set of two independent three-vectors,

$$J_i = \tfrac{1}{2}\varepsilon_{ijk} J_{jk}$$

$$K_i = J_{io} = - J_{oi}$$

in terms of which equation (20) becomes

$$\begin{aligned} [J_i , J_j] &= i\, \varepsilon_{ijk} J_k \\ [J_i , K_j] &= i\, \varepsilon_{ijk} K_k \\ [K_i , K_j] &= -i\, \varepsilon_{ijk} J_k \end{aligned} \qquad (92)$$

These commutation relations can be decoupled by defining

$$\begin{aligned} \vec{A} &= \tfrac{1}{2}[\vec{J} + i\vec{K}] \\ \vec{B} &= \tfrac{1}{2}[\vec{J} - i\vec{K}] \end{aligned} \qquad (93)$$

whose commutation relations are

$$\begin{aligned} [A_i , A_j] &= i\, \varepsilon_{ijk}\, A_k \\ [B_i , B_j] &= i\, \varepsilon_{ijk}\, B_k \\ [A_i , B_j] &= 0 \end{aligned} \qquad (94)$$

The bases of the representations of H.L.G. must therefore be labelled by the weights (A, a) and (B, b). We shall here choose, in view of the commutation relations, that $\vec{A}$ and $\vec{B}$ are represented by the generators of SU(2, C), that is, a and b take on the values $-A$ to $+A$ and $-B$ to $+B$ respectively,

$$\begin{aligned} \langle A, a; B, b | \vec{A} | A, a'; B, b' \rangle &= \delta_{bb'} \vec{J}^{(A)}_{aa'} \\ \langle A, a; B, b | \vec{B} | A, a'; B, b' \rangle &= \delta_{aa'} \vec{J}^{(B)}_{bb'} \end{aligned} \tag{95}$$

where $J^{(j)}_{\sigma\sigma'}$ are given by (71).

It is clear from (93) that for all representations (A, B) apart from (0, 0), the representations will not be unitary, because $\vec{K}$ is replaced by an anti-hermitian matrix. This means that the corresponding representation matrices listed in (65) are all different.

The simplest irreducible representations will be obtained by

$$\begin{aligned} \vec{J} &\longrightarrow \vec{J}^{(j)} \qquad \vec{K} \longrightarrow -i\,\vec{J}^{(j)} \quad : \quad (A,B) = (j,0) \\ \vec{J} &\longrightarrow \vec{J}^{(j)} \qquad \vec{K} \longrightarrow +i\,\vec{J}^{(j)} \quad : \quad (A,B) = (0,j) \end{aligned} \tag{96}$$

It follows from this, that the have representations $D^{(j)}[A]$ and $\bar{D}^{(j)}[A]$ for (j,0) and (0,j) respectively, which are related by

$$\bar{D}[A] = D^{\dagger}[A^{-1}] \,. \tag{97}$$

These matrices do not anymore operate on the spin-states as in (60) but rather on a different basis, called the Spinor Basis. This is achieved by writing the representation matrix of a little group element as

$$D^{(jo)}[\mathring{A}] = D^{(jo)}[B^{-1}_{\Lambda p \leftarrow \mathring{p}}]\, D^{(jo)}[A]\, D^{(j,o)}[B_{p \leftarrow \mathring{p}}] \tag{98}$$

and regarding the representations of the "boosts" as the transformations to the spinor basis, thus giving the operation of $D^{(jo)}[A]$ on this new basis. We shall return to this point in the next two sections.

From (97) and (68) it is easy to see that, up to a similarity transformation,

$$(A,\ B)^{*} = (B,\ A)\ . \tag{99}$$

It is also clear that

$$(A,\ B) = (A,\ 0) \times (0,\ B)\ . \tag{100}$$

As an example of this classification, we have the well known spinor $\sigma_\mu p^\mu$, which according to eq. (43) transforms with $D^{(\frac{1}{2},0)}[A] = A$ and $D^{(\frac{1}{2},\ 0)}[A] = C\, D^{(0,\frac{1}{2})}[A]C^{-1} = A^{*}$, and hence it belongs to the $(\frac{1}{2}, \frac{1}{2})$ representation. The generalisations of $\sigma . p$ to $(^{j}/2,\ ^{j}/2)$ representations are straightforward and will be given in the next Section.

According to the convention for labelling the indices of different representations, we have $\sigma_\mu \equiv \sigma_{\mu\, \alpha\dot{\beta}}$, while the spinor transforming with $\bar{A}$ and $\bar{A}^{*}$, i.e. the $(o\frac{1}{2})$ and $(0\frac{1}{2})^{*}$ representations will be $\sigma_\mu = \sigma_\mu^{\ \dot{\alpha}\beta}$, where it follows from (96), that

$$\tilde{\sigma}_\mu = (1,\ -\vec{\sigma}\,) \quad \text{if} \quad \tilde{\sigma}_\mu = (1,\ \vec{\sigma})\ .$$

From the representations (96), it is clear that $\vec{J}$ are the generators corresponding to the rotation subgroup of the homogeneous Lorentz transformations, $\vec{K}$ correspond to the pure Lorentz transformations, or, as we have called it above, boosts. We consider $D^{(1/2)}[B] = B$ as the simplest example:

$$B_{p\leftarrow \mathring{p}} \; \mathring{p}\cdot\sigma \; B^{\dagger}_{p\leftarrow\mathring{p}} = p\cdot\sigma$$
$$\mathring{p} = (m,0,0,0) \tag{101}$$

Choosing the positive square root solution of (102), we have most generally

$$B = \sqrt{\frac{p.\sigma}{m}}\; U \quad , \tag{102}$$

As in (89), the choice of $U = 1$ and $U = U(R(\hat{p}))$ correspond to the diagonalisation of the spin along z and $\hat{p}$ respectively.

With the usual parametrization of Lorentz transformations

$$p_0 = m\cosh\chi \qquad \vec{p} = \hat{p}\sinh\chi \tag{103}$$

thus

$$\sqrt{\frac{k.\sigma}{m}} = e^{-\hat{p}\cdot\frac{\vec{\sigma}}{2}\chi} = e^{-\hat{p}\cdot\vec{J}^{(\frac{1}{2})}\chi} \tag{104}$$

which is in agreement with what is given by the representation (96).

CHAPTER II

COVARIANT SCATTERING FUNCTIONS

1. Unitary Invariant S-matrix

In a scattering experiment, all the initial incoming states are given in a basis $|(\alpha)\ \mathrm{in}\rangle$ in $\mathcal{H}$, of orthogonal vectors, where (α) is the collection of all the quantum numbers labelling the state. The final outgoing states are given in another orthogonal basis $|(\beta)\ \mathrm{out}\rangle$ and therefore the transition amplitude is given by

$$\langle(\beta)\ \mathrm{out}\ |\ (\alpha)\ \mathrm{in}\rangle\ . \tag{1}$$

So that the scalar product (I.39) may be determined, the operator S is defined, such that it maps all states $|(\alpha)\ \mathrm{in}\rangle$ onto the space of states $|(\alpha)\ \mathrm{out}\rangle$:

$$\langle(\alpha)\ \mathrm{out}\ | = \langle(\alpha)\ \mathrm{in}\ |\ S\ . \tag{2}$$

From the requirement that

$$\langle(\alpha)\ \mathrm{in}\ |\ (\beta)\ \mathrm{in}\rangle = \langle(\alpha)\ \mathrm{out}\ |\ (\beta)\ \mathrm{out}\rangle$$

it follows that S is unitary, $S\,S^{+} = 1$.

We now consider a symmetry group G to each of whose elements there corresponds the unitary operation U(g) on the state vectors (Wigner's theorem). The statement that the transition probability is invariant under this symmetry is

$$|\langle\beta|S|\alpha\rangle|^2 = |\langle\beta|U^\dagger(g)\, S\, U(g)|\alpha\rangle|^2 \tag{3}$$

from which it follows that

$$\langle\beta|S|\alpha\rangle = \omega(g,\alpha,\beta)\langle\beta|U^{-1}(g)\, S\, U(g)|\alpha\rangle \tag{4}$$

where $\omega(g, \alpha, \beta)$ is a phase factor.

Using the linearity of S, one finds that $\omega(g, \alpha, \beta)$ is independent of α, β within a coherent subspace of $\mathcal{H}$, i.e. a subspace where linear superposition of states is permitted. Furthermore, this symmetry group G will be compatible with the scattering process, only if there is not transition between subspaces labelled by different eigenvalues of observables invariant under G.

Thus, within a coherent subspace we have

$$U^{-1}(g)\ S\ U(g) = \omega(g)\, S\ . \tag{5}$$

It is easy to see that $\omega(g)$ behaves like a group element

$$\omega(g_1)\ \omega(g_2) = \omega(g_1, g_2)\ . \tag{6}$$

Now in the case of relativistic invariance, these one dimensional representations of G consist of the identity representation, so that S and $U(g)$ automatically commute and hence

$$[S'[K] = S[K]\ , \tag{7}$$

where $[K]$ stands for the set of four-momentum of the external free particles.

Now our group of invariance is the I.L.G. The unitary representations of the transformations of the free particle states corresponding to the external particles of the process, is given in Sec. I, paragraph 5. An S-natrix element transforms therefore with as many $U(g)$ (or $U(g)^{+}$) as it has external incoming (outgoing) particles:

$$S[K] = \exp\left(i \sum_j k_j \cdot a\right) \prod_{\otimes i} D^{(j_i)(*)}[\mathring{A}(\Lambda)]\, S[\Lambda^{-1}K] \tag{8}$$

where $\sum k.a = 0$ by momentum conservation, and $(*)$ means complex conjugation for incoming particles only.

Before ending the paragraph, we define the R functions by

$$R = S - 1 \tag{9}$$

which transforms like

$$R[K] = \prod_{\otimes i} D^{(j_i)(*)}[\breve{A}(\Lambda)]\, R[\Lambda^{-1}K] \tag{10}$$

The little group elements here operate in a space where the third component of the spin is quantized, that is the boost operators are given by (I.56). If we took the boost operators to be given by (I.89) instead, then we would have the H-function. These are called the Helicity Amplitudes(9).

2. Spinor Amplitudes - M-functions

From the point of view of constructing solutions of scattering functions, it would be desirable to have a function which transforms simply under H.L.G. These solutions lead to the expansion of the amplitude into a set of independent scalar (invariant) amplitudes with respect to a set of bases which are the solutions themselves. We shall not, however, be concerned with any particular examples of this procedure of finding invariant amplitudes, but we are interested only in the general aspects of such scattering functions.

The M-function is defined by

$$M[K] = \prod_{\otimes i} D^{(j_i)(*)}[B_{k_i \leftarrow \mathring{p}_i}]\, R[K] \tag{11}$$

and it follows from (10) and (11) that under a Lorentz transformation $A(\Lambda)$ it transforms as

$$M[K] = \prod_{\otimes i} D^{(j_i)(*)}[A]\, M[\Lambda^{-1}K] \tag{12}$$

that is according to the $\prod_{\otimes i} (j_i, 0)^{(*)}$ representation of the H.L.G.

The simplest non-trivial example of a scattering amplitude is the elastic spin ½spin 0 scattering. Naturally there can be no process with just one spin-½ particle, as this would violate fermion conservation. The demonstration of this is very simple. From (I.89') it follows that $D^{(j)}[-A] = (-)^{2j} D^{(j)}[A]$ and from (45) we know that $\Lambda(-A) = \Lambda(A)$, therefore

$$M[K] = M[\Lambda(-1)] = (-)^{\sum_i 2j_i} \otimes D[1] M[K] = (-)^{\sum_i 2j_i} M[K] \tag{13}$$

hence, the only possible $M[K]$ are those which do not vanish identically, i.e. when $\sum_i j_i$ is integer.

Coming back to the $\frac{1}{2},0 \to \frac{1}{2},0$ example, we write (12) in this special case

$$M[K] = A \otimes A \; M[\Lambda^{-1}K] \tag{14}$$

which is in fact of the same form as (43). This equation is satisfied by the following solutions $(k_i \cdot \sigma)$, $(k_i \cdot \sigma)(k_j \cdot \tilde{\sigma})(k_\ell \cdot \sigma)$, $(k_i \cdot \sigma)(k_j \cdot \tilde{\sigma})(k_\ell \cdot \sigma)(k_m \cdot \tilde{\sigma})(k_n \cdot \sigma)$ etc....., and combinations thereof so as to give the scattering function definite signature under parity. If an R or H function has definite transformations under parity, then

$$R[K] = \pm \eta_P \; R[\tilde{K}] \; ; \; [\tilde{K}] = [K_0 - \vec{K}] \tag{15}$$

where η_P is the product of the individual parities of the process. The + sign refers to parity invariance. Using (11) we find from (15) that the transformations under parity of the M-function is

$$M[K] = \pm \eta_P \prod_{\otimes i} D^{(j_i)(*)}[BB^\dagger] M[\tilde{K}]. \tag{16}$$

It is clear that one of the most important formal apparatus in the formulation of scattering functions for any spins is the generalized spin-matrices corresponding to the operators $D^{(j)}[BB^\dagger] = D^{(j)}[(k\cdot\sigma)/m]$. We shall discuss these in the next paragraph.

3. Higher Spin Matrices

The Pauli spin matrices σ_μ can be looked upon as vectors with respect to Lorentz transformations. From equation (I.43) we can extract the following formal transformation equation:

$$D^{(\frac{1}{2})}[A] \otimes D^{(\frac{1}{2})*}[A]\ \sigma^\mu = \Lambda^\mu{}_\nu\, \sigma^\nu. \tag{17}$$

It is our purpose in this paragraph, to give the construction(6) of such matrices which transform according to the (j, j') representations of the H.L.G.:

$$D^{(j)}[A] \otimes D^{(j')}[A^*]\ \rho^{(\mu)}(j,j') = \prod_{\otimes i=1}^{2M} \Lambda^{\mu_i}{}_{\nu_i}\, \rho^{(\nu)} \tag{18}$$

where $M \equiv \max(j, j')$ and $(\mu) = \mu_1 \mu_2, \ldots, \mu_{2M}$. This is the transformation equation for a tensor of rank $2M$, which is a spin M matrix. This equation resembles (I.18) which is the transformation equation for the tensor $J_{\mu\nu}$.

The $(2j+1) \times (2j'+1)$ rectangular matrices $\rho^{(\mu)}(j,j')$ can be constructed from the σ^μ matrices by a process of spin addition with the use of Clebsch Gordan coefficients. We give below only the construction of spin-1 matrices, as the others are obtained by further straightforward spin addition:

$$\rho^{\mu\nu}_{\alpha\dot{\beta}}(j,j') = \sum [\tfrac{1}{2}\tfrac{1}{2}j]^{\gamma\gamma'}_{\alpha} \{\tfrac{1}{2}\tfrac{1}{2}j'\}^{\dot{\tau}\dot{\tau}'}_{\dot{\beta}}\ \sigma^\mu_{\gamma\dot{\tau}}\, \sigma^\nu_{\gamma'\dot{\tau}'} \tag{19}$$

where (j, j') may take on the values $(1, 1)$, $(1, 0)$, $(0, 1)$ or $(0, 0)$. The spin 1 matrix transforma as a second order tensor under $SL(2, C)$.

It follows from this construction and the following orthogonality relations of the Pauli spin matrices

$$\begin{aligned} \tfrac{1}{2}\,\sigma_{\mu\,\alpha\dot\beta}\,\tilde\sigma^{\mu\,\dot\beta'\alpha'} &= \delta_\alpha^{\ \alpha'}\,\delta_{\dot\beta}^{\ \dot\beta'} \\ \tfrac{1}{2}\,\sigma^{\mu}_{\ \alpha\dot\beta}\,\sigma_{\mu\,\alpha'\dot\beta'} &= C_{\alpha\alpha'}\,C_{\dot\beta\dot\beta'} \end{aligned} \tag{20}$$

that

$$\begin{aligned} \rho^{(\mu)}_{\alpha\dot\beta}(j,j')\,\tilde\rho_{(\mu)}^{\dot\beta'\alpha'}(j,j') &= \delta_\alpha^{\ \alpha'}\,\delta_{\dot\beta}^{\ \dot\beta'} \\ \rho^{(\mu)}_{\alpha\dot\beta}(j,j')\,\rho_{(\mu)\,\alpha'\dot\beta'}(j,j') &= D^{(j)}(C)_{\alpha\alpha'}\,D^{(j')}(C)_{\dot\beta\dot\beta'} \end{aligned} \tag{21}$$

which are the orthogonality relations of the spin-M matrices.

To verify that $P^{(\mu)}$ given by (19) satisfies the transformation equation (18) we have to use the following identity

$$[j_1 j_2 j]_\alpha^{\ \gamma\gamma'}\,[j_1 j_2 j']_\beta^{\ \tau\tau'}\,D^{(j_1)}_{\gamma\tau}[A]\,D^{(j_2)}_{\gamma'\tau'}[A] = \delta_{jj'}\,D^{(j)}_{\alpha\beta}[A] \tag{22}$$

in (18) and then use (17).

We shall denote the special cases $\rho^{(\mu)}(j,\,j)$ by $t^{(\mu)}$. These square $(2j+1)\times(2j+1)$ matrices will be particularly important in the next chapter. From their definition, (19), it follows that they are symmetric in the interchange of any of the indices in (μ) and that

$$t^{00\ldots0}_{\alpha\beta} = \delta_{\alpha\beta}\,. \tag{23}$$

Further, by using (19) and the second member of (21) we see that $t^{(\mu)}$ are traceless in any two of the indices (μ) in the following sense $g_{\mu_2\mu_1}\,t^{\mu_1\mu_2\cdots\mu_{2j}} = t_{\mu_2}^{\ \ \mu_2\mu_3\cdots\mu_{2j}} = 0$

4. M-functions for Massless Particles

In Section V, we shall have occasion to deal with M-functions for processes involving massless particles.

In paragraph 2 of this section, we pointed out that the solutions of an M-function transformation equation are combinations of objects like $D^{(j)}[k, \sigma]$. This is in fact the representations of the square of a boost operator, in particular, for the case of massive particles. If the particles involved are massless, we shall naturally have to use the corresponding boosts, which are completely different functions of the momenta from say the boost given in equation I.102). We shall not work out explicit expressions for the massless boosts till Section V, where we shall need them. Here, however, we shall only give the way in which the M-functions are modified(7) for massless particles if we start from the M-functions of the corresponding massive particles. Our description of this method(7) will be a rather inelegant demonstration for the sake of brevity.

For a spinor ξ_α transforming according to $D^{(j)}[A]$, the Weyl equation of motion

$$D^{(j)}[k \cdot \tilde{\sigma}]\ \xi = 0 \tag{24}$$

must be satisfied, and since this is also the way our M-function transforms with respect to an outgoing particle, we conclude that for each outgoing massless particle of momentum k_μ and spin j involved in the process, there should be a restriction on the M-function

$$D^{(j)}[k \cdot \tilde{\sigma}]\, M[K] = 0 \tag{25}$$

and

$$M\,[K]\;\; D^{(j)}\,[k.\,\tilde{\sigma}] \;=\; 0 \tag{26}$$

for each incoming particle.

We consider the pion-neutrino elastic scattering as an example, since we will need this result in Section V. Treating the neutrinos as massive for the moment, we enumerate four independent amplitudes, thus we would like to construct four independent spinor bases of the form $k.\sigma$ (c.f. paragraph 2 of this section) to expand the M-amplitude in terms of four scalar amplitudes $A^{i}[K]$:

$$M_{ss'}[K] = \sum_{i=1}^{4} A^{(i)}[K]\, V^{(i)}_{ss'}\,[k\cdot\sigma] \tag{27}$$

In the process, we have three independent fourmomenta k_1, k_3 and $n = k_2 - k_4$, say. A possible choice of such basis functions is

$$\begin{aligned} V^{(1)} &= k_1.\sigma \\ V^{(2)} &= k_3.\sigma \\ V^{(3)} &= n.\sigma \\ V^{(4)} &= (k_3.\sigma)(n.\tilde{\sigma})(k_1.\sigma) \end{aligned} \tag{28}$$

It is seen from (16) that these bases do not give rise to an M-function with definite parity signature, but since this is a weak process, we shall consider (28) satisfactory, and only set out to take account of the masslessness of the neutrinos. Applying (25) and (26) to (27) and (28) we find that

$$A^{(1)} = A^{(2)} = A^{(3)} = 0 \tag{29}$$

and that $M[K] = A_k^{(4)} V^{(4)} [k_j \cdot \sigma]$, (30)

that is, $\pi\nu \rightarrow \pi\nu$ has only one amplitude.

CHAPTER III

LOCAL FIELDS

1. (j, 0) and (0, j) Fields(1)

It is our purpose in this paragraph to introduce field operators $\phi^{(j)}$ and $\chi^{(j)}$, ($j \equiv$ spin), which transform according to the (j, 0) and (0, j) representations of H.L.G. respectively. We shall take it for granted that a field operator can be expanded in terms of the operators a_m and $b^{\dagger}_m$ and a set of basis functions which have the appropriate transformation properties under H.L.G. a is an annihilation operator for a particle while b+ a creation operator for an antiparticle (if there exists one different from the particle). The reason we have this combination a_m and $b^{\dagger}_m$ is that the field, which satisfies the Klein-Gordon equation, has both positive and negative energy solutions and thus, in a Fourier expansion, negative and positive frequency parts. Arguing, after Dirac, that the negative energy states must correspond to antiparticles, we conclude that the positive frequency part must be associated with the antiparticle operator, and that, because of positive frequency, it must be a creation operator. The opposite applies to the negative frequency part.

One further requirement on these fields will be that their observables satisfy causality, i.e. their commutators or anti-commutators must vanish for spacelike distances

$$\left[\psi_\alpha(x), \quad \psi_\beta(y) \right] \quad = \quad 0$$

for (x-y) spacelike. This condition cannot be satisfied except by Fermi-Bose statistics and crossing symmetry. We shall not here demonstrate this fact but will just proceed with the construction of the fields in such a way that the connection between spin and statistics is satisfied. In the next paragraph we shall have occasion to see that causality is satisfied too by these fields.

Let $a^+(\vec{p}, o)$ and $a(\vec{p}, 0)$ be the operators in Hilbert space, whose operation on the vacuum to the right and left respectively creates a single particle state of momentum $\vec{p}$ and spin projection σ . Similarly, let the operators $b^+(\vec{p}, \sigma)$ and $b(\vec{p}, \sigma)$ have the same role with respect to the antiparticles.

It is clear from this definition, that a^+ transforms, according to the single particle transformation equation (I. 64) as

$$U[A]\, a^\dagger(\vec{p},\sigma)\, U[A^{-1}] = \sum_{\sigma'} D^{(j)}_{\sigma'\sigma}[\mathring{A}^{-1}]\; a^\dagger(\overrightarrow{\Lambda p},\sigma') \tag{1}$$

and by hermitian conjugation,

$$U[A]\, a(\vec{p},\sigma)\, U[A^{-1}] = \sum_{\sigma'} D^{(j)}_{\sigma\sigma'}[\mathring{A}^{-1}]\, a(\overrightarrow{\Lambda p},\sigma') \tag{2}$$

where we have also used the unitarity of the little group representation.*

Because of the unitarity of $\mathring{A}$ and hence of $D^{(j)}[\mathring{A}_0]$, an upper (lower) dotted index is equivalent to a lower (upper), undotted index, following the remarks in Section I paragraph (6),

---

* We fix our ideas on the fields corresponding to massive particles with little groups SU(2, C).

and hence, using the raising and lowering operators of spinor indices we have the relation

$$D^{(j)}[\mathring{A}^*] = C\, D^{(j)}[\mathring{A}]\, C^{-1}. \tag{3}$$

Using the unitarity of $D[\mathring{A}]$ and (3) we can rewrite (1) as

$$U[A]\, a^{\dagger}(\vec{p},\sigma)\, U[A^{-1}] = \sum_{\sigma'} \{C\, D[\mathring{A}^{-1}]\, C^{-1}\}_{\sigma\sigma'}\, a^{\dagger}(\overrightarrow{\Lambda p},\sigma') \tag{4}$$

$b^{+}(\vec{p},\sigma)$ and $b(\vec{p},\sigma)$ transform as $a^{+}(\vec{p},\sigma)$ and $a(\vec{p},\sigma)$ respectively.

We next define operators $\alpha(\vec{p},\sigma)$ and $\beta(\vec{p},\sigma)$ as expansions in $a(\vec{p},\sigma)$ and $b^{+}(\vec{p},\sigma)$ respectively, with respect to the "spinor basis" (cf para (7), Section I).

$$\alpha(\vec{p},\sigma) = \sum_{\sigma'} D^{(j)}_{\sigma\sigma'}[B_{p\leftarrow\mathring{p}}]\, a(\vec{p},\sigma') \tag{5}$$

$$\beta(\vec{p},\sigma) = \sum_{\sigma'} \{D^{(j)}[B_{p\leftarrow\mathring{p}}]\, C^{-1}\}_{\sigma\sigma'}\, b^{\dagger}(p,\sigma') \tag{6}$$

such that they transform according to the (j, 0) representation of H.L.G.

$$U[A]\, \alpha(\vec{p},\sigma)\, U[A^{-1}] = \sum_{\sigma'} D^{(j)}_{\sigma\sigma'}[A^{-1}]\, \alpha(\overrightarrow{\Lambda p},\sigma') \tag{7}$$

$$U[A]\, \beta(\vec{p},\sigma)\, U[A^{-1}] = \sum_{\sigma'} D^{(j)}_{\sigma\sigma'}[A^{-1}]\, \beta(\overrightarrow{\Lambda p},\sigma'). \tag{8}$$

for the factor of $(-)^{2j}$ in the definition of $\beta(\vec{p}, \sigma)$, eq. (13). This factor has been arbitrarily so chosen as to have causal commutation relations between the (j, 0) type and (0, j) type fields, which so far we constructed quite independently of each other. We shall see this in the next paragraph.

Before we end this paragraph, we consider the parity (space inversion) properties of these fields.

From the well known transformation properties of single particle states under parity U(P), we can derive the following transformation law

$$U[P]\, a(\vec{p},\sigma)\, U[P^{-1}] = \eta_p\, a(-\vec{p},\sigma) \tag{14}$$

$$U[P]\, b(\vec{p},\sigma)\, U[P^{-1}] = \bar{\eta}_p\, b(-\vec{p},\sigma) \tag{15}$$

where η_p and $\bar{\eta}_p$ are phase factors,

Applying (14) and (15) to the fields $\phi^{(j)}$ and $\chi^{(j)}$ in equations (9) and (11), then changing the variable of integration $\vec{p} \leftrightarrow -\vec{p}$ we find the following transformation equations for the fields

$$\begin{aligned} U[P]\, \varphi_\sigma^{(j)}(x)\, U[P^{-1}] &= \eta_p\, \chi_\sigma^{(j)}(\tilde{x}) \\ U[P]\, \chi_\sigma^{(j)}(x)\, U[P^{-1}] &= \eta_p\, \varphi_\sigma^{(j)}(\tilde{x}) \end{aligned} \tag{16}$$

where $\tilde{x} = (x_0, -\vec{x})$. The condition on the phase factors is

$$\eta_P \; \bar{\eta}_P = (-)^{2j} \tag{17}$$

which is a consequence of locality and our desire to have the simple transformation laws (16).

In the case where we have particles which are the same as their antiparticles, we have, by setting $a(\vec{p}, \sigma) = b(\vec{p}, \sigma)$ in (19) and (11), the relations

$$\begin{aligned} \chi^{\dagger}_{\sigma}(x) &= \sum_{\sigma'} C_{\sigma\sigma'} \, \varphi_{\sigma'}(x) \\ \varphi^{\dagger}_{\sigma}(x) &= \sum_{\sigma'} (-)^{2j} \, C_{\sigma\sigma'} \, \chi_{\sigma'}(x). \end{aligned} \tag{18}$$

Finally, we remark that the Fourier transforms (9) and (11) are only relevant to fields corresponding to massive particles. As we shall not need massless particle fields in the following, we shall not introduce these, save to say that due to the transformation property of a single massless particle state, equation (I.88), which is one dimensional, it follows that $b^{+}(\vec{p}, \lambda)$ will transform just like $a(\vec{p} - \lambda)$, and $b^{+}(\vec{p}, -\lambda)$ like $a(\vec{p}, \lambda)$. Due to this fact, to satisfy causality between $\phi^{(i)}$ and $\chi^{(j)}$ fields, there is no need of the factor $(-)^{2j}$ in the analogous equation to (13). Otherwise formally everything is the same as for massive fields, however, for example the boost operators in the two cases are completely different functions of the fourmomentum.

2. Commutation Relations[10]

In Section VI, we shall be discussing the equal-time commutation relations of currents, for which we need to know the equal-time commutation relations of the fields constituting the currents. We shall give the latter in this paragraph.

We have already stated that the fields defined in the last paragraph satisfy the causality condition, and here we shall find out that indeed they do, subject to Statistics and Crossing, plus the Fermi anticommutation rules or the Bose commutation relations of the $a(\vec{p}, \sigma)$ and $b(\vec{p}, \sigma)$ operators:

$$\begin{aligned} \left[a(\vec{p},\sigma),\, a^\dagger(\vec{p}\,',\sigma') \right]_\pm &= \delta^{(3)}(\vec{p}-\vec{p}\,')\,\delta_{\sigma\sigma'} \\ \left[b(p,\sigma),\, b^\dagger(\vec{p}\,',\sigma') \right]_\pm &= \delta^{(3)}(\vec{p}-\vec{p}\,')\,\delta_{\sigma\sigma'} \end{aligned} \tag{19}$$

with all others vanishing.

The choice of the commutation rules (19) is conventional, but is not consistent with our previous definition of $a(\vec{p}, \sigma)$ and $b(\vec{p}, \sigma)$ operators. In fact, to use (19) we must make the replacement $a(\vec{p}, \sigma) \longrightarrow \sqrt{p_0}\, a(\vec{p}, \sigma)$ in (9) and (11). This is done to ensure a Lorentz invariant commutator, and we shall hereafter use this definition of creation operators throughout.

Using (19), it is straightforward to derive the commutators and anticommutators for the fields themselves:

$$\left[\varphi^{(j)}_\sigma(x), \varphi^{(j)\dagger}_{\sigma'}(y) \right]_\pm = \frac{m^{-2j}}{(2\pi)^3} \int \frac{d^3\vec{p}}{2p_0[\vec{p}]}\, \Pi^{(j)}_{\sigma\sigma'}(p_0[\vec{p}],\vec{p}) \left\{ e^{-ip\cdot(x-y)} \pm e^{ip(x-y)} \right\} \tag{20}$$

where

$$\begin{aligned} \Pi^{(j)}(p) &= m^{2j}\, D^{(j)}[B(\vec{p})]\, D^{(j)}[B^\dagger(\vec{p})] \\ &= \exp(-\hat{p}\cdot\vec{J}^{(j)}\chi) \end{aligned} \tag{21}$$

and in terms of the higher spin matrices introduced in the third paragraph of Section II, we can write

$$\Pi^{(j)}_{\sigma\sigma'}(p) = (-)^{2j}\, t^{\mu_1\mu_2\cdots\mu_{2j}}_{\sigma\sigma'}\; p_{\mu_1} p_{\mu_2}\cdots p_{\mu_{2j}} \,. \tag{22}$$

Using (22), equation (20) takes the form

$$[\varphi^{(j)}_{\sigma}(x), \varphi^{(j)\dagger}_{\sigma'}(y)]_{\pm} = i(-im)^{-2j}\, t^{\mu_1\mu_2\cdots\mu_{2j}}_{\sigma\sigma'}\; \partial_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_{2j}}\, \bar{\Delta}(x-y) \tag{23}$$

where $\bar{\Delta}$ (x-y) is the invariant function

$$\bar{\Delta}(x-y) = \frac{-i}{(2\pi)^3}\int \frac{d^3\vec{p}}{2p_0[\vec{p}]}\left\{ e^{ip\cdot x} \pm (-)^{2j} e^{-ip\cdot x}\right\} . \tag{24}$$

This satisfies causality only if the two terms have equal and opposite coefficients, that is, if we enforce the usual connection between spin and statistics:

$$\mp(-)^{2j} = 1 \;. \tag{25}$$

Equation (24) will then reduce to the usual causal invariant function Δ (x).

To see more explicitly that (24), subject to (25) satisfies causality, all we have to do is consider the equal-time case, i.e. $x_0 - y_0$, in which case, Δ (0, x - y) vanishes because the integrand becomes antisymmetric in $\vec{p}$, the variable of integration, while the limits are symmetric about zero. Now $\Delta(x-y)$ is a Lorentz scalar therefore Lorentz transformations connecting x to all points outside the light cone (space-like separation) will give $\Delta(\Lambda x) = \Delta(x)$. It follows that $\Delta(x-y) = 0$ for all $(x-y)^2 < 0$.

For $j = 0$, (23) becomes

$$[\varphi^{(0)}(x), \varphi^{(0)\dagger}(y)]_- = i\Delta(x-y), \tag{26}$$

thus spin 0 fields do satisfy causality. At first sight this is surprising because causality is to be satisfied only by field observables, that is, hermitian operators. This is in fact the case for spin 0 field operators. The reason is that as a Boson, this field has the same parity for its particle and antiparticle, and in the absence of other quantum numbers* the creation operators for particle and antiparticle are identical. Further, in this case and only in this case, the complex conjugate of the wavefunction corresponding to particle annihilation in $\phi^{(0)}$ is the same as that of antiparticle (= particle) creation in $\phi^{(0)}$, trivially, because they are both equal to one. It follows that $\phi^{(0)}$ and only $\phi^{(0)}$ is an hermitian field and satisfies the causal commutation rule (26).

For all $j \neq 0$, this procedure is inapplicable.

We shall now turn our attention to a particular case of

---

* We treat isospin and hypercharge independently and we neglect them here.

spacelike separation, namely equal-time commutation relations. We shall be needing these in Section VI.

Setting $x_o = y_o$ in (23), it is a straightforward matter to calculate the equal-time commutation relations for any j, for now the exponents in the integrand in $\Delta(x-y)$ are functions of $\vec{p}$ only, and they are either antisymmetric in $\vec{p}$ in which case the integral vanishes, or are symmetric in $\vec{p}$, in which case we get the standard representations for all orders of derivatives of δ-functions( ). This suggests that except for lightlike separations $\delta(\vec{x} - \vec{x})$, the fields at equal times commute at all spacelike separations. This is not, strictly speaking, the requirement of causality.

We give here the equal-time commutators between spin ½ and spin 1 fields, as examples

$$\left[\varphi_\alpha^{(\frac{1}{2})}(x), \varphi_\beta^{(\frac{1}{2})\dagger}(y) \right]_{+\; x_0=y_0} = m\, \delta_{\alpha\beta}\, \delta^{(3)}(\vec{x}-\vec{y}) \tag{27}$$

$$\left[\varphi_\alpha^{(1)}(x), \varphi_\beta^{(1)\dagger}(y) \right]_{-\; x_0=y_0} = -2i\, m^2\, t^{io}_{\alpha\beta}\, \partial_i \delta(\vec{x}-\vec{y}). \tag{28}$$

A spin $-\frac{3}{2}$ field equal time commutator will have both a $\delta^{(0)}(\vec{x}-\vec{y})$ and $\delta^{(2)}(\vec{x} - y)$ contribution, while for spin -2 a $\delta^{(1)}(\vec{x}-\vec{y})$ and $\delta^{(3)}(\vec{x}-\vec{y})$ and $\delta^{(5)}(\vec{x}-\vec{y})$ and so on.

It is obvious that the commutation relations between $\chi^{(j)}(x)$ and $\chi^{(j)\dagger}(y)$ will be just the same as those above, except for the replacement of $\Pi(p)$ by $\overline{\Pi}(p) = \Pi(p_o, -\vec{p})$.

In terms of the higher spin matrices defined in Section II paragraph 3, and from the representations given by (I.96), this is given by

$$\overline{\Pi}^{(j)}_{\sigma\sigma'}(p) = (-)^{2j}\, \overline{t}^{\mu_1\mu_2\cdots\mu_{2j}}_{\sigma\sigma'}\; p_{\mu_1} p_{\mu_2}\cdots p_{\mu_{2j}} \tag{29}$$

where $\overline{t}^{(\mu)}_{(j,j)}$ is the spin matrix transforming according to $D^{(j)}[\overline{A}] \times D^{(j)}[\overline{A}^*]$, or $D^{(0,j)} \otimes D^{(0,j)}$.

Finally, we consider the commutation relations between $\phi^{(j)}(x)$ and $\chi^{(j)}(y)$. It is easy to see from (9), (11) and (I.96), that this commutator is

$$[\varphi^{(j)}_{\alpha}(x),\, \chi^{(j)\dagger}_{\beta}(y)]_{\pm} = m^{2j}\, \Delta(x-y) \tag{30}$$

which is equal to zero for equal-times.

3. Vertex Functions:

In the following section (IV), we shall be concerned with some perturbation theoretic calculations, using the method of Feynman Rules[10]. For this purpose we have all the necessary formal apparatus, namely the (j, 0) and (0,j) fields and their commutators, given above in this section.

Now in the course of evaluating a Feynman diagram, we need to have the interaction Hamiltonians, which come in through Dyson's formula[14]. These Hamiltonians of interaction are required to be hermitian parity conserving Lorentz scalars, and are taken to be an invariant product of the three fields corresponding to the three particles at the vertex.

A scalar constructed from three fields with general transformation properties (j_1, j_1'), (j_2, j_2') and (j_3, j_3') is

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix}\begin{pmatrix} j_1' & j_2' & j_3' \\ \sigma_1' & \sigma_2' & \sigma_3' \end{pmatrix} \varphi^{(j_1 j_1')}_{\sigma_1 \sigma_1'}(x)\, \varphi^{(j_2 j_2')}_{\sigma_2 \sigma_2'}(x)\, \varphi^{(j_3 j_3')}_{\sigma_3 \sigma_3'}(x) \tag{31}$$

and in the case $j_1' = j_2' = j_3' = 0$ we simply have

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix} \varphi^{(j_1)}_{\sigma_1}(x)\; \varphi^{(j_2)}_{\sigma_2}(x)\; \varphi^{(j_3)}_{\sigma_3}(x) \, . \tag{32}$$

The construction of $(j, 0)$ and $(0, j)$ fields has been given above in paragraph 2 of this section, however (j, j') fields have not been discussed. The latter are easily obtained from the former remembering that $(A, 0) \otimes (0, B) = (A, B)$.

The advantage of the $(j, 0)$ and $(0, j)$ fields is, that they correspond to particles of spin j, and have $(2j+1)$ components. These are irreducible.

The advantage of (j, j') occurs in the case where $j = j'$, when, from (16) it follows that (j, j) fields transform definitely under parity. However, a disadvantage of these reducible fields is, that

$$(j, j') = (j, 0) \otimes (0, j') = (j+j') \oplus \cdots \oplus |j-j'| \tag{33}$$

corresponds to particles of spins $(j+j')$ to $|j-j'|$, and in the event of our attention being restricted to a particle of definite spin (this is the only useful case) we shall, by using such fields be faced with many components which are quite redundant. For example, when we use $\phi^{(\frac{1}{2},\frac{1}{2})}_{\sigma\sigma'} = \sigma^{\mu}_{\sigma\sigma'}\,\rho_{\mu}$ for a spin one particle, we are really dealing with two particles of spin $(\frac{1}{2}+\frac{1}{2}) = 1$ and $\frac{1}{2} - \frac{1}{2} = 0$. In this case, when we want a spin-1 particle described with three components, we have to use a Lorentz condition $\partial_{\mu}\rho_{\mu} = 0$ to get rid of one redundant component.

We will therefore prefer to deal with irreducible fields - even though they have no definite transformation under parity - so as to avoid having redundant components.

Now the scalar combination (32) is not in general hermitian. It would be hermitian only for the special case $j_1 = j_2 = j_3 = 0$. For this reason, in general (32) should be linearly combined with its hermitian conjugate.

Further, it has not got definite signature under parity as it stands. From (16) it appears that the vertex, which is a tri-linear in fields should be a combination of both $\phi^{(j)}(x)$ and $\chi^{(j)}(x)$ type fields. Neglecting all other intrinsic quantum numbers but parity, it is seen from (17) that bosons and anti-bosons can be taken to be identical since they have the same parity. Thus a boson field of type $(j, 0)$ retains its transformation character under space inversion as shown by (18), and therefore in a vertex, we need not use both types of boson fields. As for fermions, so as to have a vertex of definite space inversion properties, they must occur in pairs because of the duality of equations (16). This is in fact an expression of the conservation of Fermion number, which could be otherwise derived very simply from the transformation law of a scattering function (II.8 or 12), the homomorphism equation (I.44) and $D^{(j)}[-\overset{\circ}{A}] = (-)^{2j} D^{(j)}[\overset{\circ}{A}]$ which follows from (I.90).

Now in the invariant product (32), all the fields have the same transformation property, however, according to the argument just given, we should in general replace at least one of them by a $\chi^{(j)}(x)$ type field which, according to the contents of Section I paragraph 7, transforms differently, and hence it would not make sense to define an invariant product for them with a 3j-symbol, except if we used $C^{-1}\chi^{(j)\dagger}(x)$ for our replacement, which does transform with the same matrices as $\phi^{(j)}(x)$ does.

We have thus a way of constructing a hermitian vertex function with definite space inversion properties, provided the invariant product of $(j_1\ j_2\ j_3)$ exists. This is not the case in general, because spin-angular momentum is not conserved in relativistic interactions. It is in fact intuitively clear that since the two initial particles would in general have orbital angular momentum relative to each other, and since the representation resulting from the reduction of the representations of these two particles would be an eigenstate of the spin of the third particle, it is the sum of the orbital and intrinsic angular momenta in the initial state that must be conserved, and not either separately. We do not here discuss the reduction( ) of these representations, but just construct intuitively a vertex function such as to satisfy all our conditions, and subsequently we verify that this is the case.

What we therefore need, is a spinor object which transforms as $(\ell, 0)$, where ℓ is an integer, so that the resultant spin $\vec{j}_1 + \vec{j}_2 = \vec{j}$ can join with ℓ and j_3 in an invariant product.

That ℓ must be our integer and not a half-integer follows from the fact that the vertex involves only even numbers of Fermions. It is clear that ℓ is the relative orbital angular momentum of the two initial particles.

Before constructing the orbital angular momentum spinors, we remind ourselves of the eigenfunctions of non-relativistic orbital angular momenta, $Y^{\ell}_{m}(\theta, \phi)$, which are diagonalised with respect to the well known(4) momentum operators written in terms of the differential operators of the polar angles θ and ϕ. These are called the "Spherical Harmonics"(4). Given $Y^{(1)}_{m}(\theta, \phi)$ all higher ones can be constructed by the use of Clebsch-Gordan Coefficients (C-G.C).

We next introduce the "solid spherical harmonic" $\mathcal{Y}^{(\ell)}_{m}(\vec{p}) = r^{\ell}\, Y^{(\ell)}_{m}(\theta, \phi)$, and we define the "spherical vector" by

$$p_{\pm} = \mp \frac{1}{\sqrt{2}}(p_1 \pm i p_2) \quad ; \qquad p_0 = p_3 \tag{34}$$

In terms of the spherical coordinates, the special case of $\ell = 1$ for the solid harmonic is known to be

$$\mathcal{Y}^{(1)}_{m}(\vec{p}) = \left(\frac{3}{4\pi}\right)^{1/2} p_m \quad ; \quad m = \pm 1, 0 . \tag{35}$$

The relative orbital angular momentum of two particles with momenta $\vec{p}$ and $\vec{q}$ has then the basis

$$\begin{aligned} \mathcal{Y}^{(1)}_{m}(\vec{p},\vec{q}) &= \sum_{m_1, m_2} [111]^{m_1 m_2}_{m}\, q_{m_2}\, p_{m_1} \\ &= e_m = \frac{i}{\sqrt{2}}\,(\vec{p} \wedge \vec{q})_m \end{aligned} \tag{36}$$

where we have denoted the solid harmonic of $\ell = 1$ by e_m which is itself the spherical vector corresponding to the vector $\overline{e} = \frac{i}{2}\,\overline{p}\;\overline{q}$, Equation (36) can be written in the following form

$$e_m = \sum_{\substack{m_1 m_1' \\ m_2 m_2'}} [\tfrac{1}{2}\tfrac{1}{2}1]_m^{m_1 m_1'} [\tfrac{1}{2}\tfrac{1}{2}0]_0^{m_2 m_2'} (\vec{\sigma}\cdot\vec{p})_{m_1 m_2} (\vec{\sigma}\cdot\vec{q})_{m_1' m_2'} \cdot \quad (37)$$

What we need is the spherical vector corresponding to e_m which transforms according to the $(\ell, 0)$ representation of the H.L.G., which is[11] by analogy to (37),

$$E_m = \sum_{\substack{m_1 m_1' \\ m_2 m_2'}} [\tfrac{1}{2}\tfrac{1}{2}1]_m^{m_1 m_1'} \{\tfrac{1}{2}\tfrac{1}{2}0\}_0^{m_2 m_2'} (\sigma^\mu p_\mu)_{m_1 m_2} (\sigma^\nu q_\nu)_{m_1' m_2'} \quad (38)$$

where $\vec{E}$ can be easily calculated:

$$\vec{E}(p,q) = -p_0\vec{q} + q_0\vec{p} - i\,\vec{p}\wedge\vec{q} \cdot \quad (39)$$

Corresponding to the general $\mathcal{Y}_m^{(\ell)}(\vec{e})$ we now have $\mathcal{Y}_m^{(\ell)}(\vec{E})$, given by

$$\mathcal{Y}_m^{(\ell)}(\vec{E}) = \sum_{\substack{m_1 m_1' \\ m_2 m_2'}} [\tfrac{\ell}{2}\tfrac{\ell}{2}\ell]_m^{m_1 m_1'} \{\tfrac{\ell}{2}\tfrac{\ell}{2}0\}_0^{m_2 m_2'} \Pi_{m_1 m_2}^{(\frac{\ell}{2})}(p)\, \Pi_{m_1' m_2'}^{(\frac{\ell}{2})}(q) \cdot \quad (40)$$

The transformation property is

$$U[A]\, E_m(p,q)\, U[A^{-1}] = \sum_{m'} D_{mm'}^{(1)}[A^{-1}]\, E_m(\Lambda p, \Lambda q) \quad (41)$$

and in general

$$U[A]\, \mathcal{Y}_m^{(\ell)}(p,q)\, U[A^{-1}] = \sum_{m'} D_{mm'}^{(\ell)}[A^{-1}]\, \mathcal{Y}_m^{(\ell)}(\Lambda p, \Lambda q) . \quad (42)$$

Whereas if A was a pure rotation the $D_{mm'}^{(\ell)}$ would be unitary matrices, here in the relativistic case they are not unitary, and therefore orbital angular momentum cannot be a "good" quantum in reaction at relativistic energies.

The following properties of $\mathcal{Y}^{\ell}_{m}(\vec{E})$ are straightforward to derive from (I.99) and the symmetry properties of C-G.C.,

$$[j_1 j_2 j_3]^{m_1 m_2}_{m_3} = (-)^{j_1+j_2+j_3+2m_3} [j_1 j_2 j_3]^{-m_1,-m_2}_{-m_3} \tag{43}$$

that

$$\mathcal{Y}^{(\ell)}_{m}(p,q) = (-)^{\ell}\, \mathcal{Y}^{(\ell)}_{m}(q,p) \tag{44}$$

$$\overline{\mathcal{Y}}^{(\ell)}_{m}(p,q) = (-)^{\ell+m}\, \mathcal{Y}^{(\ell)*}_{m}(p,q) \tag{45}$$

where $\overline{\mathcal{Y}}^{(\ell)}_{m}$ is the space-inverted $\mathcal{Y}^{(\ell)}_{m}$.

We are now in a position to construct the vertex functions, having defined the relativistic (spinor) orbital angular momentum functions in terms of momenta. What we need of course are vertex functions in configuration space, so that we replace all momenta k_μ by $-i\partial_\mu$. The operation of these derivatives on the Fourier transforms of the fields involved in the vertices gives us back the relativistic solid spherical harmonics in momentum space.

The most general vertex function will contain two different Fermions of spins j_1 and j_2, and one Boson of spin j_3. As we have mentioned previously Bosons and antiBosons can be regarded as being identical because they have the same parity (space inversion phase), treating all other quantum numbers as "internal" and completely independent of space-time symmetry. For this reason corresponding to the spin-j_3 particle, there will be only a $(j_3, 0)$ field and no $(0, j_3)$ field in the vertex, since in this case the latter is connected to the former by (18). The scalar in (32) will now take the form

$$H(x)=\sum_{\ell,s} G_{\ell,s}(x)\,[j_1 j_2 s]^{\sigma_1\sigma_2}_{\sigma}\begin{pmatrix} s & \ell & j_3 \\ \sigma & m & \sigma_3\end{pmatrix}\left\{\left[C^{-1}\chi^{(j_2)\dagger}(x)\right]_{\sigma_2}\varphi^{(j_1)}_{\sigma_1}(x) + \begin{bmatrix} j_1 \\ \sigma_1\end{bmatrix} \rightleftharpoons \begin{bmatrix} j_2 \\ \sigma_2\end{bmatrix}\right\} \mathcal{Y}^{(\ell)}_m\left[-i\partial^{(1)}, -i\partial^{(2)}\right]\varphi^{(j_3)}_{\sigma_3}(x) \tag{46}$$

where the arguments of $\mathcal{Y}^{\ell}_m$, $\partial^{(1)}$ and $\partial^{(2)}$ refer, respectively, to operation on $\phi^{(j_1)}(x)$ and $\chi^{(j_2)}(x)$, and the function in curly brackets has been made symmetric in $(j_1, \sigma_1) \rightleftharpoons (j_2, \sigma_2)$ which is done to give the vertex function definite parity transformation properties. The summations of ℓ and s, the orbital and spin angular momenta run over the values permitted by the vector addition equation $\vec{j}_3 = \vec{s} + \vec{\ell}$, and, in the event that parity is conserved, some of these values will be excluded. After going to momentum space each term will be multiplied by a function $G_{\ell,s}(k^2)$, which is assumed to be an analytic function of the one scalar k^2, which is the only independent scalar constructed from the two independent fourmomenta involved in the (vertex) three particle process. These functions are called form factors and are in fact the Fourier transforms of smearing functions in configuration space. There are, of course, in general several such independent couplings for any one vertex. For example, for the $N\bar{N}\gamma$ vertex, angular momentum and parity conservation dictate two form factors, $\{s = 1, \ \ell = 0\}$ and $\{s = 1, \ \ell = 2\}$ while for $N\bar{N}\gamma$ there are three form factors with $\{s = 1, \ \ell = 0\}$,

$\{s = 1 \quad \ell = 2\}$ and $\{s = 2, \quad \ell = 2\}$ respectively.

The function $H(x)$ does not have definite transformation properties. Using space inversion laws (16) for the fields, subject to (17) for the case of the boson of spin-j_3, and the relations (43) and (45), we find that

$$U[P]\, H(x)\, U[P^{-1}] = \eta_P\, H^{\dagger}(\tilde{x}) \tag{47}$$

where $\eta_p = \eta_{j_1}\, \eta_{j_2}\, n_{j_3}$. It is now clear that the parity conserving hermitian interaction Hamiltonian is:

$$H_I(x) = H(x) + H^{\dagger}(x) \tag{48}$$

$$H_I(x) = iH(x) - iH^{\dagger}(x) \tag{49}$$

(48) being appropriate when $\eta_p = +1$ and (49) when $\eta_p = -1$.

We shall finish this paragraph by making some remarks on the vertex functions $H_I(x)$.

In (46) the bilinear in the field is symmetric between $(j_2, \sigma_2) \rightleftharpoons (j_1, \sigma_1)$. The second term here is only relevant in the case of j_1 and j_2 referring to different fermions. When $j_1 = j_2$ and ϕ and $\chi^{\dagger}$ refer to the particle (antiparticle) and antiparticle (particle), then the second term is quite unnecessary. An example of this is the $\bar{N}N\pi$ vertex, which has a unique coupling with orbital angular momentum $\ell = 0$ and $\eta_p = -1$. In this case (46) simplifies very considerably and gives

$$H_I(x) \simeq i\left(\chi^{\dagger}_{\sigma}(x)\, \varphi_{\sigma}(x) - \varphi^{\dagger}_{\sigma}(x)\, \chi_{\sigma}(x)\right) \varphi^{(0)}(x) \tag{50}$$

having used the identity $\sqrt{2}\,[\tfrac{1}{2}\tfrac{1}{2}0]^{\alpha\beta}_{0} = C^{-1}_{\alpha\beta}$. Using the following familiar notation

$$\psi_\alpha(x) = \begin{bmatrix} \varphi_\alpha(x) \\ \chi_\alpha(x) \end{bmatrix} \;;\; \gamma_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix};\; \gamma_i = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix};\; \gamma_5 = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix} \tag{51}$$

we can rewrite (50) as

$$H_1(x) \simeq i\,\psi^\dagger(x)\,\gamma_0\gamma_5\,\psi(x)\,\varphi^{(0)}(x) = i\,\bar{\psi}(x)\gamma_5\,\psi(x)\,\varphi^{(0)}(x) \tag{52}$$

which is the familiar vertex expression in Dirac formalism. This particular vertex happens to be the only one which in the conventional formalism is made up entirely of the irreducible (H.L.G.) fields, namely (½, 0) and (0, 0).

In vertices which involve only Bosons, it is also unnecessary to retain the term $(j_2, \sigma_2) \rightleftharpoons (j_1, \sigma_1)$ in (46), because in this case the $\chi(x)$ are related to $\phi(x)$ through (18), so in any case the vertex need only be composed of $\phi(x)$-type fields.

CHAPTER IV

POLE ANALYSIS FOR THE $\pi\rho$ SYSTEM.

1. Introduction

In this section, we shall give an explicit application of the formalism described in the first paragraph and the apparatus developed in the third paragraph of the last section, by actually working out Feynman amplitudes from Rules derived[10] from the above-mentioned formalism.

We shall be concerned with the $\pi\rho$ system, and in particular, the structure of the A2 and ω resonances as generated by u-channel exchanges of themselves and A1 and π mesons.

According to the bootstrap[12,13] idea, the force in the s-channel, that is, the force binding the $\pi\rho$ resonance in the direct channel, arises from the processes (i.e. singularities in the amplitude) in the u (or t) channel, that is all particle and resonance exchanges in the crossed channels. A particularly interesting case is, that, when a particle (resonance) is bound predominantly by its own cross-channel exchange, in which case the amplitude which is dominated by the pole in the s variable at the mass of the resonance in question, is given completely (predominantly) by the cross-channel exchange amplitude due to crossing symmetry. Now this last amplitude can be evaluated by Feynman Rules, and is not unitary. We shall not here describe how this amplitude is unitarised, save to say that it is done by a process of iteration[12,13]. Then two requirements are forced on this amplitude. Firstly, that it should have a pole in s at the $(\text{mass})^2$ of the resonance, and second, that the coupling constants should be related to the residue of this pole. Now both these parameters are exactly the same ones as used for input, while calculating the cross-channel Born amplitude, and so we have two equations and two unknowns, from which we can evaluate these parameters self-consistently. This limited discussion concludes our exposition of the bootstrap (self-consistent) method of calculating masses and coupling constants, because we shall not actually carry out this process in the following.

What we shall be concerned with will be the <u>conditions</u> that must be satisfied, so that such a calculation may be carried through. The bootstrap condition is, that the cross-channel amplitude with a pole in u (or t) at the $(\text{mass})^2$ of the resonance in question, when projected into the channel appropriate to this resonance in the s-channel with respect to all quantum numbers, be dominant as compared to the projection coming from amplitudes dominated by cross-channel poles corresponding to all other possible resonances.

So far we have restricted our attention to a resonance, whose binding is mostly due to a force, itself generated by the same resonance. This of course does not necessarily have to be the case for a self consistent calculation of masses and coupling constants. In principle, all that has to be satisfied is that there should be an equal number of parameters and self-consistency equations, that is, the cross-channel exchanges of resonances $A_1, A_2, \ldots\ldots, A_n$ be the dominant contributions of the direct channel resonances $A_1, A_2, \ldots\ldots, A_n$. A well known example of this, is the reciprocal bootstrap[12b] situation between the πN resonances $N_{\frac{1}{2}\frac{1}{2}}$ and $N^*_{\frac{3}{2}\frac{3}{2}}$. Here the N^* turns out to be predominantly bound by the u-channel exchange of N, while N turns out to be bound by the u-channel exchange of N^*.

We shall investigate in this section, whether the bootstrap conditions are satisfied for a reciprocal bootstrap between A_2 and ω meson-resonances, that is, we shall carry out the s-channel projections into the A_2 and ω channels, of the u-channel amplitudes arising from the exchanges of A_2, ω, and A_1 and π mesons.

Our first task will be to evaluate the u-channel pole-amplitudes. To this end, we shall introduce the Feynman Rules for any spins[10] in paragraph 2. In paragraph 3 we shall calculate the coupling constants, needed in the evaluation of the amplitudes, from the experimental widths, and, in paragraph 4, these amplitudes will be projected into the A2 and ω channels respectively, and the bootstrap condition will be discussed.

All experimental data used : masses, resonance widths, decay momenta and all discrete quantum numbers are taken from the U.C.R.L.-8030-Part I, August 1965.

2. Feynman Rules[10]

For the evaluation of Feynman diagrams, there are three formal objects needed - interaction Hamiltonians $H_I(x)$ for each vertex at x, wave functions for external particles and propagators for internal particles.

We start by assuming that the S-matrix can be calculated from Dyson's formula(14)

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \cdots d^4x_n \, T\{ H_I(x_1) \cdots H_I(x_n) \} \tag{1}$$

where $H_I(x_1)$ is the interaction Hamiltonian, given in paragraph 3 of Section III. As far as Feynman rules are concerned, these H_I need a few further technical qualifications:

Considering a three particle process, with a Fermion and Antifermion (of the same particle, $j_1 = j_2 = j$) incoming and an outgoing Boson, we can write down the vertex-function in momentum space by

$$\int d^4x \langle \vec{k}, \sigma_3 | H_I(x) | p_1 \sigma_1 ; q, \sigma_2 \rangle = \int d^4x \, \langle 0 | a_3(\vec{k}, \sigma_3) [H_1(x) + H^{\dagger}(x)] \, a_1^{\dagger}(\vec{p}, \sigma_1) \, b_1^{\dagger}(\vec{q}, \sigma_2) | 0 \rangle \tag{2}$$

After using the C.R. (III.19) for a and b operators, in the term H(x), it is $\phi^{(j_1)}_{\sigma_1}(x)$ whose Fourier transform will be expanded in terms of p and $\chi^{(j_1)\dagger}_{\sigma_2}(x)$ in terms of q ; whereas in the term $H^{+}(x)$ the converse will be the case. Thus, after carrying out the d^4x integration, the solid spherical harmonic in the $H(x)$ term will be a function of the momenta p, q, that is $\mathcal{Y}^{(\ell)}_m(p, q)$ while the one in the term $H^{+}(x)$ will be $\mathcal{Y}^{(\ell)}_m(q,p)$. In the event $j_1 \neq j_2$, the first term in eq. (III.46) will have a factor of $\mathcal{Y}^{(\ell)}_m(p,q)$ and the second term $\mathcal{Y}^{(\ell)}_m(q,p)$. In the case of Bosons this consideration does not apply, for we assume particle and antiparticle to be the same.

A further point to be mentioned is, that $\mathcal{Y}^{(\ell)}_{m}$ can be a function of any of the two independent momenta out of the three fourmomenta p, q and k (p+q = k) involved in the vertex. This can be seen directly by considering the basic relativistic solid harmonic $\vec{E}(p,k)$, eq. (III.39), in which we replace k by p+q with the result that $\vec{E}(p, p+q) = \vec{E}(p, q)$. We shall make use of this point in the next paragraph in simplifying certain calculations.

Finally we write down explicitly the vertex expression we shall be needing in the $\pi\rho \to \pi\rho$ example, that is the vertex involving $\pi\rho$ coupled to a spin-j Boson (e.g. A2, A1, ω, π) which would be an internal particle:

$$H_I(x) = \sum_{\ell} g_\ell \begin{pmatrix} 1 & j & \ell \\ \sigma_1 & \sigma & m \end{pmatrix} \varphi^{(1)}_{\sigma_1}(x)\, \varphi^{(j)}_{\sigma}(x)\, \mathcal{Y}^{(\ell)}_{m}(\partial^{(1)}, \partial^{(j)})\, \varphi^{(0)} + \eta_\rho (h.c) \tag{3}$$

We note here that the $\partial^{(j)}$ in $\mathcal{Y}^{(\ell)}_{m}$ does not operate on one of the incoming, that is, free particles, but to the internal particle of spin-j. This will give rise to $\mathcal{Y}^{(\ell)}_{m}(p, k)$, which, as we have pointed out, is equal to $\mathcal{Y}^{(\ell)}_{m}(p,q)$.

This concludes our discussion of the vertex function in momentum space, rendering it ready for application to Feynman diagrams.

Next we shall introduce the wave functions corresponding to external particles. To this end we shall rewrite the Fourier expansions of the fields completely explicitly, having taken account of the redefinition of the creation operators in paragraph 2 of Section III. It is then a trivial matter to read off the wave-functions as simply the coefficients of the operators in the integrands of these Fourier transforms:

$$\varphi^{(j)}_{\sigma}(x) = (2\pi)^{-3/2} \int \frac{d^3\vec{p}}{\sqrt{2\omega(\vec{p})}} \left[D^{(j)}_{\sigma\sigma'}[B(\vec{p})]\, a(\vec{p},\sigma')\, e^{-ipx} + \{ D^{(j)}[B(\vec{p})] C^{-1} \}_{\sigma\sigma'} b^{\dagger}(\vec{p},\sigma')\, e^{ipx} \right] \tag{4}$$

$$\chi^{(j)}_{\sigma}(x) = (2\pi)^{-3/2} \int \frac{d^3\vec{p}}{\sqrt{2\omega(\vec{p})}} \left[D^{(j)}_{\sigma\sigma'}[B(-\vec{p})]\, a(\vec{p},\sigma')\, e^{-ipx} + (-)^{2j} \{ D^{(j)}[B(-\vec{p})] C^{-1} \}_{\sigma\sigma'} b^{\dagger}(\vec{p},\sigma')\, e^{ipx} \right] \tag{5}$$

For example, the wavefunction for a particle destroyed by $\phi_{\sigma_o}^{(j)}(x)$ is $(2\pi)^{-3/2} \left[2\omega(\vec{p}) \right]^{\frac{1}{2}} D^{(j)}_{\sigma\sigma'} \left[B(\vec{p}) \right] \exp(-ip.x)$, and obviously its complex conjugate is the wavefunction for a particle created by $\phi^{(j)\dagger}_{\sigma}(x)$. Here σ' is the spin projection along the z-axis. If we wish to calculate the helicity amplitudes, then we must replace these boosts by the helicity boosts given in (I.89) and (I.103). In the following paragraphs of this section we shall be concerned with the explicit evaluation of amplitudes for the $\pi\omega$ system. We shall list below all the wavefunctions we shall need. These are such wavefunctions arising only from ϕ-type fields, for in an all-Boson problem, we need only use ϕ-type (or χ-type) fields. The following wavefunctions are derived from those in expansions (5) by using (I.89) and the identity

$$D^{(j)}[R(\hat{q})]\ \hat{q}\cdot\vec{J}^{(j)}\ D^{(j)}[R^{\dagger}(\hat{q})] = J_3^{(j)} \tag{6}$$

(this is just a special case of (I.43) when A is unitary).

These wavefunctions for a (Bose) particle of momentum $\vec{p}$

and helicity λ are

created by ϕ : $V_\sigma(x;\vec{p},\lambda) = \frac{(2\pi)^{-3/2}}{(2\omega)^{1/2}} (-)^{-j+\lambda} D^{(j)}_{\sigma,-\lambda}[R(\hat{p})]\, m^{-\lambda} (\omega+|\vec{p}|)^\lambda e^{-ipx}$

(7)

annihilated by ϕ : $U_\sigma(x;\vec{p},\lambda) = \frac{(2\pi)^{-3/2}}{(2\omega)^{1/2}} D^{(j)}_{\sigma\lambda}[R(\hat{p})]\, m^{\lambda} (\omega+|\vec{p}|)^\lambda e^{ipx}$

with the wavefunction for creation and annihilation of the particle by $\phi^\dagger$ given by $V^*_\sigma(x;\vec{p},\lambda)$ and $U^*_\sigma(x;\vec{p},\lambda)$ respectively.

We shall in fact only need the above wavefunctions for ρ-mesons, with $j = 1$.

It is clear that the wavefunction of a particle is given by the boost operator after the x-integrations are carried out in (1). It follows thus from the definition of the M-functions (II.11) in Section II, that the propagators in momentum space correspond exactly (up to some normalisations) to the spinorial amplitudes M, and , in an S-matrix philosophy[5,13] we should therefore require that the propagators transform according to the representations of H.L.G. Here we use irreducible fields $(j, 0)$ and $(0, j)$, so that our propagators should in particular transform according to irreducible representations of the H.L.G.

The propagator derived from Wick's Theorem[14] is given by

$$\langle 0|T\{\varphi_\sigma(x)\varphi^\dagger_{\sigma'}(y)\}|0\rangle = \theta(x-y)\langle 0|\varphi_\sigma(x)\varphi^\dagger_{\sigma'}(y)|0\rangle + (-)^{2j}\langle 0|\varphi^\dagger_{\sigma'}(y)\varphi_\sigma(x)|0\rangle\theta(y-x)$$

(8)

Substituting (3) into (5), and by using the conventional normalisation of states (I.37) we obtain

$$\langle 0|T\{\varphi^{(j)}_{\sigma}(x)\,\varphi^{(j)\dagger}_{\sigma'}(y)\}|0\rangle$$

$$=(2\pi)^{-3} m^{-2j}\int\frac{d^3\vec{p}}{2p_0}\left[\theta(x-y)\Pi^{(j)}_{\sigma\sigma'}(p)e^{-ip(x-y)}+(-)^{2j}\theta(y-x)\Pi^{(j)}_{\sigma\sigma'}(p)\,e^{ip(x-y)}\right]$$

$$= S^{(j)}_{\sigma\sigma'}(x-y)+\Sigma^{(j)}_{\sigma\sigma'}(x-y) \tag{9}$$

where, in terms of (III.) $S_{\sigma\sigma'}$ is given by

$$S^{(j)}_{\sigma\sigma'}(x-y) = i(-im)^{-2j}\, t^{\mu_1\mu_2\cdots\mu_{2j}}_{\sigma\sigma'}\,\partial_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_{2j}}\,\Delta_F(x-y) \tag{10}$$

Δ_F being the propagator function for a spin-0 particle

$$\Delta_F(x) = (2\pi)^{-3}\; i\int\frac{d^3\vec{p}}{2p_0[\vec{p}]}\left[\theta(x)e^{-ipx}+\theta(-x)e^{ipx}\right] \tag{11}$$

and $\Sigma_{\sigma\sigma'}$ are the terms arising from having commuted the $\partial_{\mu_1}\cdots\cdots\partial_{\mu_{2j}}$ from the right of the θ-functions to the left. For $j = 0$ $\Sigma^{(0)}$ is equal to zero and we just have Δ_F. For $j = ½$, $\Sigma^{(1/2)}_{\sigma\sigma'}$ vanishes again, because the action of the derivation on the θ- function in both terms of (11) produces δ-functions $\delta(x_0)$ and $-\delta(x_0)$ respectively, and after the integration over dx_0 we end up with a definite integral of $d^3\vec{p}$ between limits $-\infty$ and $+\infty$ and an integrand antisymmetric in $\vec{p}$. This integral vanishes.

For $j \geqslant 1$ not all integrals contributing to $\Sigma^{(j)}_{\sigma\sigma'}$ vanish and their contributions are in fact noncovariant in the sense of eq. (II.12), because the covariant derivative ∂_μ operating on $\theta(x_0)$ gives only $\partial_0\,\theta(x_0) = \delta(x_0)$ which is no more covariant. For $j \geqslant 1$ higher order derivatives of δ-functions come into play.

According to our requirement of the covariance of the propagator, we shall retain only the covariant part $S^{(j)}_{\sigma\sigma'}(x-y)$ in the Wick propagator (9) and we shall neglect the non-covariant parts $\Sigma^{(s)}_{\sigma\sigma'}(x-y)$. The disappearance of these terms may be formally arranged for by adding suitable terms to the Hamiltonian[10].

Thus the momentum space propagators coming into the Feynman diagrams directly are

$$\begin{aligned} S^{(j)}_{\sigma\sigma'}(q) &= \int d^4x \; e^{-iqx} \; S_{\sigma\sigma'}(x) \\ &= -i\,(-m)^{-2j}\, \Pi^{(j)}_{\sigma\sigma'}(q)/(q^2-m^2+i\varepsilon) \end{aligned} \tag{12}$$

having used the well known[14] momentum representations of the invariant causal functions.

The momentum space propagators $\langle 0|\, T\{\phi(x)\;\chi^{+}(y)\}\,|0\rangle$ arising from Wick's Theorem are proportional to $m^{-2j}\Delta_F(x-y)$.

The propagators are therefore given by the $\Pi^{(j)}(p)$ matrices given in (III.21) whose explicit calculation[10] is a straightforward matter. We list below those for $j = 1$ and $j = 2$ which we need in our example

$$\Pi^{(1)}(p) = p^2 + 2(\vec{p}\cdot\vec{J}^{(1)})(\vec{p}\cdot\vec{J}^{(1)} - p_0)$$

$$\begin{aligned} \Pi^{(2)}(p) &= (p^2)^2 + 2p^2\,(\vec{p}\cdot\vec{J}^{(2)})(\vec{p}\cdot\vec{J}^{(2)} - p_0) \\ &\quad + \tfrac{2}{3}(\vec{p}\cdot\vec{J}^{(2)})\left[(\vec{p}\cdot\vec{J}^{(2)}) - \vec{p}^{\,2}\right]\left[\vec{p}\cdot\vec{J}^{(2)} - 2p_0\right]. \end{aligned} \tag{13}$$

3. The Coupling Constants

In the last paragraph, vertices, propagators and wavefunctions are presented explicitly, and the only further information we need for evaluating amplitudes numerically are the coupling constants associated with each vertex, apart of course from the values of the masses of the particles involved. These coupling strengths are given by the experimental decay widths, and we shall calculate them below.

For a resonance of spin-j and mass m_j, decaying at rest into the two particles ρ and π of mass m_ρ and m_π and decay momentum $\vec{p}$ and $-\vec{p}$, the formula obtained[15] for the width Γ ((lifetime)$^{-1}$), is given, after carrying out the phase-space integral, summing over the final spin-states and averaging over the initial spin states:

$$\Gamma = \frac{2}{(2j+1)} \frac{p}{m_j} \sum_f \sum_i |\mathcal{M}_{fi}|^2 \tag{14}$$

where

$$\mathcal{M}_{fi} = \sqrt{p_\pi^0} \sqrt{p_\rho^0}\, R_{fi} \sqrt{p_j^0} \tag{15}$$

R_{fi} being the amplitude calculated by the Feynman Rules given in the last paragraph.

The $|\mathcal{M}_{fi}|^2$ includes the factor $g^2_{j\pi\rho}$, the coupling constant corresponding to vertex coupling the spin-j resonance with π and ρ. We shall proceed with the evaluation of these absolute squared amplitudes and summing them as in (14).

Using (3) for the vertex and the wavefunctions in the field expansion (4)xm^j, we can find $\mathcal{M}_{fi}$. Then, from (I.99) and (I.97) we have

$$D^{(j)}[B(\vec{p})]^* = C^{(j)}\,\bar{D}^{(j)}[B(\vec{p})]\,C^{(j)-1} \tag{16}$$

by means of which, and the definition of $\mathbb{II}$-matrices, we obtain the expression

$$\sum_f \sum_i |\mathcal{M}_{fi}|^2 = \sum_{\substack{\sigma\sigma' \\ \sigma_1\sigma_1'}} 2 g_{j\pi\rho} \Big\{ \begin{pmatrix} 1 & j & \ell \\ \sigma_1 & \sigma & m \end{pmatrix} \begin{pmatrix} 1 & j & \ell \\ \sigma_1' & \sigma' & m' \end{pmatrix} \mathbb{II}^{(1)}_{\sigma_1\sigma_1'}(p_\rho)\, \mathbb{II}^{(j)}_{\sigma\sigma'}(p) + \\ + \begin{pmatrix} 1 & j & \ell \\ -\sigma & \sigma & m \end{pmatrix} \begin{pmatrix} 1 & j & \ell \\ \sigma_1' & -\sigma & m' \end{pmatrix} \eta_\rho (-)^{1+j+\sigma+\sigma_1'} m_\rho^2 m_j^{2j} \Big\} \mathcal{Y}^{(\ell)}_m(p_\rho p_j) \mathcal{Y}^{(\ell)}_{m'}(p_\rho p_j) \tag{17}$$

We note that since the process is planar (in particular linear) the $\mathcal{Y}^{(\ell)}_m$ are real. We have already used this property in (17). We now go on to make further simplifications which arise from the special kinematics.

Choosing the o to be produced along the z-axis we have $p_j = (m_j, \vec{0})$ and $p_o = (p_o^{\,o}, 0, 0, p)$. It follows that

$$\mathbb{II}^{(\ell/2)}_{\alpha\alpha'}(p_j) = (-)^\ell \; t^{00\ldots0}_{\alpha\alpha'} (m_j)^\ell = (-)^\ell \delta_{\alpha\alpha'} m_j^\ell \tag{18}$$

and

$$\mathbb{II}^{(\ell/2)}_{\beta\beta'}(p_\rho) = (-)^\ell \; t^{\mu_1\ldots\mu_\ell}_{\beta\beta'} p_{\rho\mu_1} \cdots p_{\rho\mu_\ell} \tag{19}$$

where the $t^{(\mu)}_{\beta\beta'}$ is constructed (c.f. (II.19)) entirely out of σ_0 and σ_3 which are both diagonal, and hence $t^{(\mu)}_{\beta\beta'}$ is also diagonal and

$$\Pi^{(\ell_2)}_{\beta\beta'}(p_\rho) = 0 \quad \text{if} \quad \beta \neq \beta'. \tag{20}$$

It follows from (18) that

$$\mathcal{Y}^{(\ell)}_m(p_\rho, p_j) \sim \left[\tfrac{\ell}{2}\ \tfrac{\ell}{2}\ \ell\right]^{-\beta'\beta}_m \Pi^{(\ell/2)}_{\beta\beta'}(p_\rho) \tag{21}$$

which subject to (20) has a non-vanishing contribution only for $m = 0$. In particular, we shall need $\mathcal{Y}^{(2)}_o(p_\rho, p_j)$, $\mathcal{Y}^{(1)}_o(p_\rho, p_j)$ in our example, which are given explicitly:

$$\mathcal{Y}^{(2)}_o(p_\rho, p_j) = \frac{4}{\sqrt{6}} m_j^2\, p^2 \ ; \ \mathcal{Y}^{(1)}_o(p_\rho, p_j) = -\sqrt{2}\, m_j\, p. \tag{22}$$

With these simplifications, the sums in (17) can be carried out either trivially or very easily.

We are interested in the following four couplings

$$\begin{array}{lllll} \pi\pi\rho: & j = 0 & \eta = -1 & \ell = 1 & \\ \pi\rho\omega: & j = 1 & \eta = -1 & \ell = 1 & \\ \pi\rho A_1: & j = 1 & \eta = +1 & \ell = 0,\ 2 & (\ell = 2 \text{ neglected}) \\ \pi\rho A_2: & j = 2 & \eta = +1 & \ell = 2 & \end{array} \tag{23}$$

and denoting (17) by $\sum(j)$, we arrive, after an easy and straightforward calculation at the following results:

$$\begin{aligned} \sum(\pi) &= g^2_{\pi\pi\rho}\ \frac{8}{3}\ m_\rho^{\ 4}\ p^2 \\ \sum(\omega) &= g^2_{\pi\omega\rho}\ \frac{8}{3}\ m_\omega^{\ 3}\ p(p^2 + m^2) \\ \sum(A1) &= g^2_{\pi\rho A1}\ \frac{8}{3}\ m_{A1}^{\ 2}\ (4p^2 + 7m^2) \\ \sum(A2) &= g^2_{\pi\rho A2}\ \frac{32}{15}\ m_{A2}^{\ 8}\ p^4(p^2 + m^2) \end{aligned} \tag{24}$$

Substituting (24) into (14) and using the experimental data for p and Γ, as well as the masses, we find

$$
\begin{aligned}
g^2_{\rho\pi\pi} &= 0.2610 & \qquad g^2_{A1\rho\pi} &= 0.5650 \\
g^2_{\omega\rho\pi} &= 0.1165 & \qquad g^2_{A2\rho\pi} &= 1.820 \quad .
\end{aligned}
\tag{25}
$$

We end this paragraph with two final notes.

In (17), the wave function used comes from fields given in (4), with an additional factor m^j. This is unnecessary here, but is included in the general formalism, to avoid the inverse of masses appearing, in view of applications to massless particles too.

Finally, in the third member of (23), there are really two couplings for the $\pi\rho A1$ vertex, while we have neglected the $\ell = 2$ coupling. This approximation would be unjustifiable for processes at high energies, because the harmonics corresponding to high (ℓ) contain greater powers of momenta, and so their relative importance rises with energy. The above calculations are of course at threshold, and hence our neglect of $\ell = 2$, compared with $\ell = 0$, is justified.

4. Projection and Results

The calculation of the helicity amplitudes(9) for the u-channel exchanges of π, ω, A1 and A2 is straightforward by the use of the vertex (3), the helicity wavefunctions (7), the propagators (12) and the coupling constants, with the assignments (23). We shall not write out these u-channel amplitudes explicitly, because it would be cumbersome and is not instructive.

Crossing symmetry is a well known property of Feynman amplitudes. By applying the substitution Rule - that is, replacing the u-channel four momentum of each particle which is incoming (outgoing) in both channels by its s-channel four momentum, and, by replacing the u-channel four momentum of each particle which is outgoing (incoming) in the other, by the negative of its four momentum in the s-channel - we obtain the above mentioned u-channel amplitudes, the s-channel amplitudes $H_\lambda(s, \cos\theta)$ with θ the scattering angle. All that is left to do now is to project the required total angular momentum projection, corresponding to the spin, J, of the s-channel resonance whose binding we are investigating. This definite angular momentum amplitude is given by(9)

$$H^J_{(\lambda)}(s) = \frac{1}{2}\int_{-1}^{+1} d(\cos\theta)\; H_{(\lambda)}(s,\cos\theta)\, d^J_{\mu\nu}(\theta)$$

$$\mu = \lambda_3 - \lambda_4 \qquad \nu = \lambda_1 - \lambda_2 . \qquad (26)$$

Further, the amplitude we require must have definite parity, namely the parity of the s-channel resonance. To this end, we consider the transformation property of two-particle (spin j_1, j_2)

helicity states[9] under parity,

$$U[P]\,|J,M;\lambda_1\lambda_2\rangle = \eta_1\eta_2(-)^{J-j_1-j_2}\,|J,M;-\lambda_1,-\lambda_2\rangle \tag{27}$$

where J is the total angular momentum, M its projection. For the $\pi\omega$ system, the following combinations are eigenstates of P

$$|\underset{\sim}{\pm}\rangle = \tfrac{1}{2}\left[\,|J,M;1,0\rangle \pm |J,M;-1,0\rangle\right]$$

$$|\underset{\sim}{0}\rangle = \quad |J,M;00\rangle \tag{28}$$

the eigenvalues being $(-)^{\ell} = (-)^{J-1}$, $(-)^{J}$, $(-)^{J-1}$ respectively. This basis diagonalizes the S-matrix, so that we may write it in the form

$$S_\ell = e^{2i\delta_\ell} \tag{29}$$

$$S_\ell = 1 + if_\ell \ , \tag{30}$$

similar to the matrix elements of S between eigenstates of total angular momentum

$$S^J_{fi} = \langle JM\,;\,f\,|\,S\,|\,JM\,;\,i\rangle \tag{31}$$

$$S^J_{fi} = \delta_{fi} + i\,H^J_{fi} \tag{32}$$

where $f \equiv \lambda_3, \lambda_4$ and $i \equiv \lambda_1, \lambda_2$.

From the inverse of (28), and using (29), (31), gives us

$$
\begin{aligned}
S^{J}_{++} &= \tfrac{1}{2}\left[e^{2i\delta_{J-1}} + e^{2i\delta_{J}} \right] \\
S^{J}_{+-} &= \tfrac{1}{2}\left[e^{2i\delta_{J-1}} - e^{2i\delta_{J}} \right] \\
S^{J}_{00} &= e^{2i\delta_{J-1}}
\end{aligned}
\tag{33}
$$

Finally, substituting (23) into (32), then using (29) and (5) we have

$$
\begin{aligned}
f_{J\pm1} &= \tfrac{1}{2}\left[H^{J}_{++} + H^{J}_{+-} \right] \\
f_{J} &= \tfrac{1}{2}\left[H^{J}_{++} - H^{J}_{+-} \right] \\
f_{J\pm1} &= H^{J}_{00} .
\end{aligned}
\tag{34}
$$

It is clear that ℓ is in fact the orbital angular momentum of the two particle system. In this sense, both ω and A2 are coupled uniquely to $\pi\rho$, that is, through a single value of ℓ each, which happens to be $\ell = 1 = J$ and $\ell = 2 = J$ respectively, that is, the relevant amplitude in both these cases would be f_J, given by the second member of (34). Thus, substitution of (26) into f_J in (34) completes the process of angular momentum and parity projection.

So far we have ignored one good quantum number, the total isotopic-spin( ). Each u-channel amplitude has a definite total isospin, equal to that of the exchange resonance, and the projection of this amplitude onto the s-channel amplitude with definite total isospin equal to that of the s-channel resonance is obtained by use of the isospin crossing matrix( ) for $\underline{1} \otimes \underline{1}$ $\underline{1} \otimes \underline{1}$:

$$\begin{bmatrix} \frac{1}{3} & 1 & \frac{5}{3} \\ \frac{1}{3} & \frac{1}{2} & -\frac{5}{6} \\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \end{bmatrix}$$

This completes the process of projection.

The integration in (26) and the summation over the repeated indices in the vertices and propagators were carried out on a computer, the only input being the incoming energy squared, s, which was set to a value of 8.2×10^5 $(\mathrm{MeV})^2$ while at the threshold itself, s is 8.15×10^5 $(\mathrm{MeV})^2$.

Assuming that a partial wave amplitude such as our $f_\ell(s)$ tends to zero as $k^{2\ell}$ near threshold, k being the incoming C.M. three-momentum, we divide our results by k^{2J} to cancel out this effect.

The results of the calculation of the amplitudes and their projections into the appropriate amplitudes contributing to the resonances in question are listed below.

u-exchange amplitude of resonance R, projected into the direct-channel amplitude of resonance R' :

R		π	ω	A1	A2
R' :	ω	-6.984×10^{-9}	-3.048×10^{-8}	-2.804×10^{-2}	-2.352×10^{-9}
	A2	$+2.092\times10^{-4}$	-1.672×10^{-4}	-5.434	5.048×10^{-5}

The energy units used in the calculations were 10^{-3} MeV.

A variety of conclusions may be drawn from this table. The most striking is that ω is not bound at all by forces arising from any of these exchanges. While A2 seems to be bound somewhat by forces due to its own exchange, this does not give any indication of a bootstrap, because there is a larger contribution to its binding from π exchange. This of course does not exclude the possibility of a bootstrap among π and A2, to investigate which possibility, we would have to project the above amplitudes onto the π direct channel. This, however, is probably not very fruitful, since in any case we could not have a reciprocal bootstrap actually, as the contribution from π to A2 is not really dominant with respect to the contribution of A2 to A2 itself, in which event the problem loses the desirable simplification present in the reciprocal case. The final conclusion is that neither ω, nor A2 seem to be suitable for selfconsistent calculations, either by bootstrap or reciprocal bootstrap.

CHAPTER V

CROSSING RELATIONS AND SUPERCONVERGENCE

In paragraph 1 of this section we shall derive crossing relations for C.M. helicity amplitudes for processes involving massless particles, and in paragraph 2 we shall give an application of these crossing relations to the superconvergence of helicity amplitudes.

1. The Crossing Relations:

We start by assuming that scattering amplitudes are crossing symmetric. This follows too from the Feynman Rules presented in the last section, as we shall demonstrate below.

As pointed out in Section II, the scattering amplitude may be transformed into the M-function by the operation of boosts (eq. (II.11)). The M-function can be expanded in the form $M_{(s)} = \sum_i A^i Y^i_{(s)}$, where A^i are independent Lorentz scalars and functions of the invariants formed from the energy momentum vectors in the scattering process, and $Y^i_{(s)}$ are Lorentz covariant spinor bases, of the form $D(k.\sigma)$. In the special case of a two particle scattering process, the invariants are

$$\begin{aligned} s &= (k_1 + k_2)^2 \\ t &= (k_1 - k_3)^2 \qquad\qquad (1) \\ u &= (k_1 - k_4)^2 \end{aligned}$$

where $[K] = [k_4, k_3, k_2, k_1]$ are the four energy momentum vectors, with k_1 and k_2 incoming.

Now in any scattering diagram, $\sum_{i}^{n} k_i = \sum_{f}^{m} k_f$ (n initial and m final particles), so that any incoming (outgoing) external line with momentum k can be replaced by an outgoing (incoming) line with momentum $-k$. This change will alter the meaning of the invariants. For example in the 2-particle process, if this change is applied to k_2, then $s = (k_1 + k_2)^2 \quad s = (q_1 - q_2)^2$, where we have changed $[K]$ into $[Q]$ to denote the change of the physical meaning of s from incoming energy to momentum transfer. Nevertheless, the M-function will be the same function of the invariants. Moreover, because of its simple transformation property (II.12) under H.L.G., the M-function will be the same function in all Lorentz frames. Naturally, this kind of change $(k \longrightarrow -k)$ can only be done by going through the complex values of k, and this can be done only if the M-function is an analytic function. This in fact will be an assumption we make in general, and in particular for Feynman amplitudes, it turns out to be the case anyway.

Turning now to the physical amplitudes $R[K]$ or $H[K]$, we investigate the change they will suffer under crossing (analytic continuation), by examining (II.11). The boost operators given by $BB^{+} = (k.\sigma)/m$ have square roots of k, and we shall choose the positive square root for the purpose of analytic continuation. These boosts are in fact wavefunctions, as given in (III.7,8,9) in Section III, and after the crossing operation, we see that a wave function $D[B(\vec{k})]$ corresponding to an incoming particle $(\text{III. }9)^{\dagger}$ becomes $D[B(-\vec{k})]$, which, through (IV.16) is the wavefunction for the outgoing antiparticle. This means that the physical scattering amplitude too remains the same function of the momenta under crossing, that is inverting an external line. Crossing, therefore, may be regarded as a consequence of locality, which was responsible for the way in which the Fourier expansions of the fields were made in Section III, paragraph 1.

For any physical scattering function $f[K]$, the statement of Crossing is therefore:

$$f^{(s)}[K] \quad = \quad f^{(t)}[\overline{K}] \tag{2}$$

where $f^{(s)}_{[K]}$ is the amplitude evaluated in the physical region of the variable s (e.g. s is the incoming total energy) and $f^{(t)}[Q]$ the amplitude evaluated in the physical region of t (e.g. $t = (q_1 + q_3)^2$ is the total incoming energy), and $[\overline{K}]$ is the set $[K]$ with the sign of the fourmomentum of the inverted particle reversed. In the particular case of two-particle scattering, with which we shall be concerned in the following, $[\overline{K}] = [k_4, -k_3, -k_2, k_1]$ which is our definition of the t-channel.

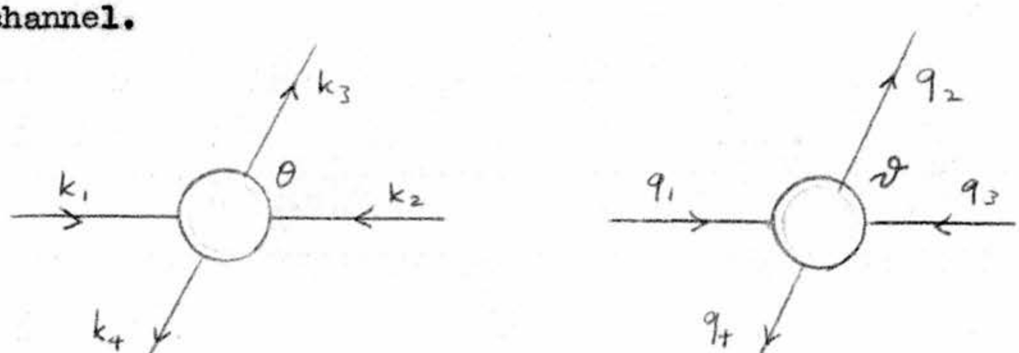

Fig. 1.

We are now in a position to derive the crossing relations( ) for C.M. helicity amplitudes. Given $H\left\{{t \atop (\lambda)}\right\}[Q]$ we must find $H\left\{{s \atop (\lambda)}\right\}[K]$ where now $[K]$ is the set of momenta in the C.M. of the t-channel. $(\lambda) \equiv (\lambda_4, \lambda_3, \lambda_2, \lambda_1)$. The equation (2) may not be directly applied in this case, because, in the general case where the masses $m_1 \neq m_2 \neq m_3 \neq m_4$ $(m_i \geqslant 0)$ the C.M. frame of 1 and 2 (s-channel) is in motion with respect to the C.M. frame of 1 and 3 (t-channel), while the two amplitudes in (2) belong to the same frame of reference. Another way of seeing this, is to examine the transformation properties of R or H physical amplitudes and M spinor amplitudes, (II.10) and (II.12) respectively. Whereas M transforms according to H.L.G. and hence is the same in all frames, R and H do not, and are defined with respect to a special frame, namely the C.M. frame in the case of the helicity amplitudes H.

It follows from the above that before analytically continuing $H\left\{{t \atop (\lambda)}\right\}[Q]$ in the $[Q]$, we must bring this scattering function from the t-channel C.M. into the frame in which the C.M. of the s-channel is at rest, by a Real Lorentz transformation parametrised by the "angle" (imaginary) χ. As the process considered is planar (2 particles $\longrightarrow$ 2-particles) it is sufficient to apply a Real Lorentz transformation along the z-axis to affect this change of frame. The "angle" for such a Lorentz transformation is given by

$$\begin{aligned} q_1'^{0} + q_3'^{0} &= \sqrt{t}\ \cosh\chi \\ |\vec{q}_1' + \vec{q}_3'| &= \sqrt{t}\ \sinh\chi \end{aligned} \qquad (3)$$

taking $[Q]$ to $[Q']$. This is essentially a boost taking the t-channel C.M., $(q_1^0 + q_3^0\ ,\ \vec{q}_1 + \vec{q}_3) = (\sqrt{t}, 0, 0, 0)$

into a frame $(q_1'^0 + q_3'^0, \vec{q}_1' + \vec{q}_3')$, in which the process is designated by the set of real fourmomenta $[Q'] = [q_4', q_3', q_2', q_1']$, and which happens to be referred to the frame in which the C.M. of the s-channel is at rest. The new function is given by (II.10) to be

$$H^{(t)}[Q'] = \prod_{\otimes n} \prod_{\otimes m} D^{(j_n)*}[\mathring{A}(q_n', \chi)] \otimes D^{(j_m)}[\mathring{A}(q_m', \chi)]\, H^{(t)}[Q] \tag{4}$$

$$m = 2, 4 \quad ; \quad n = 1, 3$$

Analytically continuing (4), that is $[Q'] \longrightarrow [\bar{K}]$, we get $H^{(t)}[\bar{K}]$ whence, using (2) we get

$$H^{(s)}[K] = \prod_{\otimes n} \prod_{\otimes m} D^{(j_n)*}[\mathring{A}(\bar{k}_n, \bar{\chi})] \otimes D^{(j_m)}[\mathring{A}(\bar{k}_m, \bar{\chi})]\, H^{(t)}[Q] \tag{5}$$

where $\bar{k}_i$ is given by $[\bar{K}] = [k_4, -k_3, -k_2, k_1] = [\bar{k}_4, \bar{k}_3, \bar{k}_2, \bar{k}_1]$ and $\bar{\chi} = \chi[\bar{K}]$ (c.f. $\chi = \chi[K]$). Note that $[Q']$ has been replaced by $[\bar{K}]$, while no explicit change is made to $[Q]$, because as we shall list below (eq.s 13) $[Q]$ is expressed in terms of the invariant $t = (q_1+q_3)^2 = (q_1' + q_3')^2$, so that the effect of analytic continuation on $[Q]$ (t) is the replacement $t = (q_1 + q_3)^2 \longrightarrow t = (k_1 - k_3)^2$.

The crossing relations of the helicity amplitudes for processes with $m_1 \neq m_2 \neq m_3 \neq m_4$ $(m_i \geqslant 0)$ is given by (5), provided we use the little group elements by (I.63) and (I.72) for $m_i \neq 0$ and $m_i = 0$ respectively.

Alternatively, we could have continued analytically in the variables $[Q]$ to $[K']$; $[K'] = [\bar{Q}] = [q_4, -q_3, -q_2, q_1]$, which is the set of fourmomenta in the s-channel physical region and in the C.M. frame of the t-channel. From (2) we have $H^{(s)}[K'] = H^{(t)}[Q]$, to which then we would apply the Lorentz transformation which takes $[K']$ into $[K]$, where the C.M. of the s-channel is at rest. This is the inverse of the boost which takes $(k_1{}^o + k_2{}^o, \vec{k}_1 + \vec{k}_2) = (\sqrt{s}, 0, 0, 0)$ to $(k_1'^o + k_2'^o, \vec{k}_1' + \vec{k}_2')$, and is parametrised by the "angle" given by

$$\begin{aligned} k_1'^o + k_2'^o &= \sqrt{s} \cosh \chi' \\ |\vec{k}_1' + \vec{k}_2'| &= \sqrt{s} \sinh \chi' \end{aligned} \tag{6}$$

These two procedures are equivalent as we shall verify later. They consist of a Lorentz transformation (change of frame) and analytic continuation and the converse respectively. This is in fact equivalent to a Complex Lorentz transformation. We demonstrate this here, for the first alternative:

In the t-channel physical region, we have, from (I.43)

$$A^{-1}(\chi)\ q'.\sigma\ A^{-1}(\chi) = q.\sigma \tag{7}$$

where we have taken $A^+ = A$, which is due to our freedom to restrict to the x-z plane. After continuation in the $[Q']$, we get

$$(\pm)\ A^{-1}(\bar{\chi})\ k.\sigma\ A^{-1}(\bar{\chi}) = q.\sigma \tag{8}$$

$(\pm)$ corresponding to particles 1, 4 and 2,3 respectively. Taking for simplicity the special value $\mathrm{ch}\bar{\chi} = (k_1^o - k_3^o)/\sqrt{t} = 0$

which is the case for elastic reactions, we find from (8) that now

$$q = \pm\left(ik\sin\frac{\theta}{2},\ \tfrac{1}{2}k\sin\theta - iE\cos\frac{\theta}{2},\ 0,\ iE\sin\frac{\theta}{2} + k\cos^2\frac{\theta}{2}\right) \tag{9}$$

where E is the energy, k the magnitude of the three-momentum of the particle, θ the polar angle of scattering in the s-channel C.M., and $A(\bar{\chi})$ is given by

$$\begin{aligned} A(\bar{\chi}) &= \mathrm{ch}\,\frac{\bar{\chi}}{2} + \vec{\sigma}\cdot\hat{p}\,\mathrm{sh}\,\frac{\bar{\chi}}{2} \\ \hat{p} &\parallel (\vec{k}_1 - \vec{k}_3) \end{aligned} \tag{10}$$

Thus, if we regarded the crossing relations as being the elements of a group of transformations, then this will be a group of complex transformations which take real vectors into complex vectors. Furthermore, this group must be the group of complex Lorentz transformations, because the norm of the vector q in (9) is the invariant mass

$$q^2 = m^2 . \tag{11}$$

The length of the "space" part of the vector in (9) is

$$|\vec{q}| = i\sqrt{E^2 - k^2\cos^2\frac{\theta}{2}} \tag{12}$$

Before proceeding to evaluate the crossing relations explicitly, and branching off into the details of the massive and massless elements of these relations separately, we list(19) the sets of C.M. fourmomenta [K] and [Q] in terms of the invariants s and t respectively:

$$
\begin{array}{ll}
(a) & (b) \\
k_1^0 = (s + m_1^2 - m_2^2)/2\sqrt{s} & q_1^0 = (t + m_1^2 - m_3^2)/2\sqrt{t} \\
k_2^0 = (s - m_1^2 + m_2^2)/2\sqrt{s} & q_3^0 = (t - m_1^2 + m_3^2)/2\sqrt{t} \\
k_3^0 = (s + m_3^2 - m_4^2)/2\sqrt{s} & q_2^0 = (t + m_2^2 - m_4^2)/2\sqrt{t} \\
k_4^0 = (s - m_3^2 + m_4^2)/2\sqrt{s} & q_4^0 = (t - m_2^2 + m_4^2)/2\sqrt{t} \\
k_i = \left[(s-(m_1+m_2)^2)(s-(m_1-m_2)^2)\right]^{1/2}/2\sqrt{s} & q_i = \left[(t-(m_1+m_3)^2)(t-(m_1-m_3)^2)\right]^{1/2}/2\sqrt{t} \\
k_f = \left[(s-(m_3+m_4)^2)(s-(m_3-m_4)^2)\right]^{1/2}/2\sqrt{s} & q_f = \left[(t-(m_2+m_4)^2)(t-(m_2-m_4)^2)\right]^{1/2}/2\sqrt{t}
\end{array}
\tag{13}
$$

where k_i, k_f are the magnitudes of the initial and final C.M. three-momenta, and similarly q_i and q_f.

The explicit evaluation of the crossing relations is essentially the explicit evaluation of the little group elements in (5), which we shall consider separately in the massive and massless particle cases.

In (I.58), the little group element corresponding to a Lorentz transformation A, is defined in terms of A and the boost matrices as

$$
\mathring{A}(k,\Lambda) = B^{-1}_{\Lambda k \leftarrow \mathring{p}}\, A(\Lambda)\, B_{k \leftarrow \mathring{p}} \tag{14}
$$

in the two-dimensional complex (basic) representation. It is by explicit parametrization of A and evaluation of B(k) that we shall find $\mathring{A}(k, \Lambda)$.

(i) <u>Massive particle</u>[17,18]:

The helicity boost for this case is already given by (I.102) and (I.104), but a simpler expression for a helicity boost is

$$B_{q\leftarrow\mathring{p}} = U(\hat{q})\, Z_{q\leftarrow\mathring{p}} = U(\hat{q})\, e^{-J_3\xi} = U(\hat{q})\, e^{-\lambda\xi} \tag{15}$$

$$\begin{aligned} q^0 &= m\cosh\xi \\ |\vec{q}| &= m\sinh\xi \end{aligned} \tag{15'}$$

where Z is a pure Z-direction boost, and q is a t-channel four-momentum.

It is now possible to evaluate $\mathring{A}$ using (15) for B and (10), with $\hat{p} = \hat{z}$ for A, however, it turns out that in this case the manipulations involved simplify somewhat if we use the Lorentz transformation matrices $\Lambda^{\mu}{}_{\nu}$ themselves, which are given by (I.44) in terms of (15) and (10) respectively. Thus we find that (suppressing $\mu(\nu) = 2^{nd}$ row (column))

$$\Lambda[A] = \begin{bmatrix} ch\chi & 0 & sh\chi \\ 0 & 1 & 0 \\ sh\chi & 0 & ch\chi \end{bmatrix} \quad \Lambda[Z] = \begin{bmatrix} ch\xi & 0 & sh\xi \\ 0 & 1 & 0 \\ sh\xi & 0 & ch\xi \end{bmatrix} \quad \Lambda[U] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \tag{16}$$

and by using these in the corresponding equation to (14) we find that

$$\Lambda[\mathring{A}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & \sin\omega \\ 0 & -\sin\omega & \cos\omega \end{bmatrix} \tag{17}$$

that is $\mathring{A}$ is a rotation about the y-axis through an angle given by

$$\cos\omega = \cos\vartheta\cos\vartheta' + \mathrm{ch}\,\chi\,\sin\vartheta\sin\vartheta' \tag{18}$$

which can be shown to be

$$\mathrm{ch}\,\chi = \mathrm{ch}\,\xi\,\mathrm{ch}\,\xi' - \cos\omega\,\mathrm{sh}\,\xi\,\mathrm{sh}\,\xi' . \tag{19}$$

All primed parameters pertain to $q' = \Lambda(\chi)q$. Using (15') and (19) we have

$$\cos\omega_n = \frac{q_n^\circ q_n'^\circ - m_n^2\,\mathrm{ch}\,\chi[Q]}{q_n q_n'} \tag{20}$$

where now we have started labelling by the particles n = 1,2,3,4, and the notation is just that in (13). This is the angle of the "Wigner Rotation". $\overset{\circ}{A}$ itself is just equal to $d^{(1/2)}(\omega)$ where $d^{(j)}$ are the reduced rotation matrices( ).

We now rewrite (4) by means of the parameter (real angle) ω

$$H^{(t)}[Q'] = \prod_{\otimes n}\prod_{\otimes m} d^{(j_n)*}(\omega_n) \otimes d^{(j_m)}(\omega_m)\, H^{(t)}[Q] \tag{21}$$

where we have replaced the rotation matrices $D^{(j)}$ by the reduced rotation matrices $d^{(j)}$, because the Wigner rotation is only around the y-axis.

After analytic continuation, $[Q'] \longrightarrow [K]$, we have (5) in the form

$$H^{(s)}[K] = \prod_{\otimes n}\prod_{\otimes m} d^{(j_n)*}(\bar{\omega}_n) \otimes d^{(j_m)}(\bar{\omega}_m)\, H^{(t)}[Q] \tag{22}$$

where ω_n is given by

$$\cos\bar{\omega}_n = \frac{\pm q_n^0 k_n^0 - m_n^2 \,\mathrm{ch}\,\bar{\chi}}{q_n k_n} \tag{23}$$

$\pm$ belonging to $n = 1,4$ and $n = 2,3$ respectively.

Alternatively, had we chosen the second path of crossing, that is, continuing in $[Q] \to [K'] = [\bar{Q}]$ first, and then transforming with $\Lambda^{-1}[\chi']$, given by (6), $[K'] \longrightarrow [K]$ we would have obtained, for the Wigner rotations

$$\cos\omega_n' = \frac{k_n^0 k_n'^0 - m_n^2 \,\mathrm{ch}\,\chi'}{k_n k_n'} \tag{24}$$

and after continuation

$$\cos\bar{\omega}_n' = \frac{\pm k_n^0 q_n^0 - m_n^2 \,\mathrm{ch}\,\bar{\chi}'}{k_n q_n} \tag{25}$$

Using (13) in (3) and (6) we find that

$$\begin{aligned} \mathrm{ch}\,\bar{\chi} &= \frac{k_1^0 - k_3^0}{\sqrt{t}} = \frac{m_1^2 - m_2^2 - m_3^2 + m_4^2}{2\sqrt{ts}} \\ \mathrm{ch}\,\bar{\chi}' &= \frac{q_1^0 - q_2^0}{\sqrt{s}} = \frac{m_1^2 - m_2^2 - m_3^2 + m_4^2}{2\sqrt{st}} \end{aligned} \tag{26}$$

from which it follows that (23) and (25) are identical, and that the two paths of crossing are completely equivalent.

Using (13) in (24) or (26), it can be shown after some straightforward algebra that

$$-1 \leq \cos \overline{\omega} \leq 1 \tag{27}$$

that is, the crossing relations are real.

In the πN elastic reaction example, taking particles 1, 3 to be nucleons and 2,4 to be pions, the relation given between $(H^{(s)}_{\frac{1}{2}0,\frac{1}{2}0}[K]\ ,\ H^{(s)}_{\frac{1}{2}0,-\frac{1}{2}0}[K])$ and $(H^{(t)}_{\frac{1}{2}\frac{1}{2},00}[Q],\ H^{t}_{\frac{1}{2}-\frac{1}{2},00}[Q])$ is just the same as the relation between the Frazer-Fulco( ) amplitudes (G_{++}, G_{+-}) and (F_{++}, F_{+-}).

(ii) Massless particle:

The elements of the little group for this case were discussed in Section I, paragraph 6. Here we start by evaluating the boost operators( ) explicitly. In contrast to the procedure in (i) of this paragraph, it will be more convenient for us to work in the SL(2, C) space. In the 2×2 complex (spin-½) representation, from

$$\begin{gathered} B_{q\leftarrow \mathring{p}}\ \mathring{p}\cdot\sigma\ B^{\dagger}_{q\leftarrow\mathring{p}} = q\cdot\sigma \\ \mathring{p} = (\omega, 0, 0, \omega) \end{gathered} \tag{28}$$

and by using the unimodularity of B, we find the general form of B to be

$$B = \begin{bmatrix} \sqrt{\dfrac{q_0+q_3}{2\omega}}\ e^{i\varphi/2} & B_{12} \\ \dfrac{q_1+iq_2}{\sqrt{2\omega(q_0+q_3)}}\ e^{i\varphi/2} & \sqrt{\dfrac{2\omega}{q_0+q_3}}\ e^{-i\varphi/2} + B_{12}\ \dfrac{q_1+iq_2}{q_0+q_3} \end{bmatrix} \tag{29}$$

where ϕ is a real angle and B_{12} is as yet arbitrary. To remove this arbitrariness, we seek a special solution B_o of (28), which is given by

$$B_{q \leftarrow \overset{o}{p}} = B^o_{q \leftarrow \overset{o}{p}} \overset{o}{A}{}^{o} \tag{30}$$

That B^o is a solution can be seen easily by substituting (30) into (28) and using (I.62).

Taking the arbitrary phase in (29) to be the same as that in (I.72), and using (I.72) in (30), we find that B^o will be independent of the arbitrary complex number z occurring in (I.72) for such a special value of the arbitrary element B_{12}, that the ensuing B^o is

$$B^o_{q \leftarrow \overset{o}{p}} = \begin{bmatrix} \sqrt{\dfrac{q_0 + q_3}{2\omega}} & 0 \\ \dfrac{q_1 + i q_2}{\sqrt{2\omega(q_0 + q_3)}} & \sqrt{\dfrac{2\omega}{q_0 + q_3}} \end{bmatrix} \tag{31}$$

In our case of 2-particle planar scattering, we can set $q_2 = 0$.

If we choose a special form of $\overset{o}{A}$

$$\overset{o}{A} \begin{bmatrix} 1 & \frac{\omega}{q} \tan\frac{\vartheta}{2} \\ 0 & 0 \end{bmatrix} \tag{32}$$

where ν is given in Fig. 1, then (30) can be rewritten as

$$B_{q \leftarrow \mathring{p}} = U(\hat{q})\, Z_q \tag{33}$$

where now Z_q is the pure z-direction boost, obtained by setting $q_3 = q$ and $q_1 = q_2 = 0$ in (31). (33) is in fact the limit( ) obtained from (I.102) as $(q/m) \longrightarrow \infty$, and this is the boost we shall use in (14).

In terms of (34) and (10) with $p = z$, we find for the one-dimensional little group element

$$\mathring{A}[Q, Q'] = \exp \tfrac{i}{2}[\psi - i \ln (q/q')] = e^{i\varphi/2} \tag{34}$$

and
$$D^{(j)}[\mathring{A}] = \exp i j \varphi$$

with
$$\cos \frac{\psi}{2} = \cos \frac{\vartheta' - \vartheta}{2}\, ch \frac{\chi}{2}$$
$$\sin \frac{\psi}{2} = -i \cos \frac{\vartheta' + \vartheta}{2}\, sh \frac{\chi}{2} \tag{35}$$

and all representations $D^{(j)}[\mathring{A}]$ corresponding to other spins given by (I.90). The one-dimensional nature of these representations is manifested quite independently through the following condition arising from the masslessness of the particle in the equation $q' = \Lambda q$:

$$\cos\vartheta \cos\vartheta' + ch\chi \sin\vartheta \sin\vartheta' = 1 \tag{36}$$

which is the analog of (19).

After the analytic continuation of (5), we obtain for $\overset{\circ}{A}[Q, \overline{K}]$ from (34), (35), by using the condition (36) and $\sin\vartheta'[Q, \overline{K}] = 0$:

$$\overset{\circ}{A}[Q, \overline{K}] = \sqrt{q/k}\left[ch\,\frac{\overline{\chi}}{2} + sh\,\frac{\overline{\chi}}{2}\right] \tag{37}$$

$$\overset{\circ}{A}{}^{*}[Q, \overline{K}] = \sqrt{k/q}\left[ch\,\frac{\overline{\chi}}{2} - sh\,\frac{\overline{\chi}}{2}\right] \tag{37'}$$

$\sin\vartheta'$ as a function of $[Q]$ and $[\overline{K}]$, that is after crossing, is equal to zero because, before the crossing, it was the angle between $\vec{q}_1{}'$ and $\vec{q}_2{}'$ (see Fig. 1) and after it becomes the angle between $\vec{k}_1$ and $-\vec{k}_2$, which are antiparallel.

In each of the following types of processes

(a) four massless particles

(b) three massless and one massive

(c) two massless and two massive unequal

(d) one massless and three massive unequal

it can be seen by using (13) and (3) that (37) is of the following form

$$\overset{\circ}{A}[Q, \overline{K}] = \sqrt{Z} + \sqrt{Z^{*}} \tag{38}$$

where Z is some complex function of $[\overline{K}]$ and $[Q]$. Clearly (38) is a real form, so that the crossing relations for massless particle processes are real as expected.

Finally, we remark on the matrix structure of the crossing relation elements corresponding to massless particles. Since it is a c-number (one-dimensional), it must be taken as $\delta_\lambda^{\lambda'}$ where λ is the helicity of that particle in the s-channel and λ' the helicity in the t-channel. This only applies to particles which transform under the same representation of the rotation groups in both channels, that is, are both in (out) going in their respective channels. Particles which are in (out) going in one channel and out (in) going in the other have different representations (complex conjugate) which are unitarily equivalent, so that in this case they would be related by $D^{(j)}\ C$ and $D^{(j)}\ C^{-1}$ respectively, given in (I.69) and not by 1.

Before we give any application, we shall give a check of these crossing relations in the pion-neutrino elastic reaction. From paragraph 4 of section II, we know that there is only one invariant amplitude in this case and the spinor basis for the M-function is

$$Y^{(s)}[K] = (k_3 \cdot \sigma)(n \cdot \tilde{\sigma})(k_1 \cdot \sigma)$$
$$n = k_2 + k_4 \tag{39}$$

in the s-channel and therefore, in the t-channel

$$Y^{(t)}[Q] = (-q_3 \cdot \sigma)(n' \cdot \tilde{\sigma})(q_1 \cdot \sigma)$$
$$n' = -q_2 + q_4 \tag{40}$$

where we have the same definition of s and t as previously, and particles 1 and 3 are the neutrinos. The helicity basis[6] may be obtained by using (II.11) and (33), and with the choice of

$\omega = k_o$ (c.f. (29)) we find them to be

$$Z^{(s)} = 2(s-m^2)\sqrt{[st+(s-m^2)]/s} \tag{41}$$

$$Z^{(t)} = 2 \qquad \sqrt{[st+(s-m^2)]t} \tag{42}$$

Denoting the crossing relation formally by $\mathbb{X}$, and substituting $H^{(s)} = A\,Z^{(s)}$ and $H^{(t)} = A\,Z^{(t)}$ into $H^{(s)} = \mathbb{X}\,H^{(t)}$ we have $Z^{(s)} = \mathbb{X}\,Z^{(t)}$ which gives us

$$\overline{X} = \frac{s-m^2}{\sqrt{-ts}} \tag{43}$$

This is in fact exactly what is obtained for $\mathbb{X}$ by substituting the appropriate little group elements for the neutrinos from (37) into (5), having used (13) and (3) resulting in $\mathrm{ch}\,\overline{\chi} = 0$ and $\mathrm{sh}\,\overline{\chi} = -i$ leading to $\mathrm{ch}\,\frac{\overline{\chi}}{2} = \frac{1}{\sqrt{2}}$ and $\mathrm{sh}\,\frac{\overline{\chi}}{2} = -\frac{i}{\sqrt{2}}$.

2. Superconvergent Amplitudes

A scattering function $f(s, t)$ which is analytic in s and has a suitable asymptotic s-behaviour satisfies the dispersion relation

$$f(s,t) = \int ds' \frac{\mathrm{Im}\, f(s',t)}{s'-s} \tag{44}$$

for fixed t. If now $f(s, t)$ vanishes faster than $s^{-(n+1)}$ then it is called a superconvergent amplitude and satisfies the following relations

$$\begin{aligned} &\int \mathrm{Im}\, f(s,t)\, ds = 0 \\ &\cdots\cdots\cdots \\ &\int \mathrm{Im}\, f(s,t)\, s^n\, ds = 0 \end{aligned} \tag{45}$$

By approximating $f(s, t)$ with various pole approximations, and saturating (45) with these pole intermediate states, one ends up with sum rules between the form factors or coupling constants pertaining to the couplings of these intermediate states to the external particles of the process.

Naturally, for a process involving spinning particles, there will be several such amplitudes, and hence several sum rules like (45), and, the higher the spins the more this tendency will be. In fact, it is only for spinning amplitudes that we can have superconvergence, because some of the amplitudes happen to suffer an extra power of convergence with asymptotic s, in addition to the convergence of the amplitudes otherwise known from unitarity[20] for example. This extra convergence arises from kinematical factors occurring in, say the $Y_{(s)}$, the basis functions in the expansion of M into invariant amplitudes, e.g. (II.27). For example, some $A^{(i)}$ (or combinations) may converge faster than others. These are not, however, the amplitudes we shall apply to superconvergence relations in this paragraph. We shall instead be interested in writing down such relations for helicity amplitudes. The superconvergence of helicity amplitudes for massive processes has been considered[21] before, and here we shall extend it to the case where the process involves massless particles too.

The essential point in writing dispersion relations for helicity amplitudes is the observation[22] that each term in the expansion* into invariant amplitudes of $H^{(t)}_{(\lambda)}(s, t)$ contains a factor

$$\chi_{\mu,\nu}(\vartheta) = \left(\cos\frac{\vartheta}{2}\right)^{|\nu+\mu|}\left(\sin\frac{\vartheta}{2}\right)^{|\nu-\mu|}$$

$$\mu = \lambda_1 - \lambda_2 \quad , \quad \nu = \lambda_3 - \lambda_4 \tag{46}$$

Considering this, it has been argued[23] that the following amplitudes

$$h^{(t)}_{(\lambda)}(s,t) = H^{(t)}_{(\lambda)}(s,t) \,/\, \chi_{\mu,\nu}(\vartheta) \tag{47}$$

will be analytic in s, because it is analytic in $\cos\vartheta$, by having removed the $\chi_{\mu\nu}$ factors which include square roots of $\cos\vartheta$. We may therefore write dispersion relations like (44) in s

---

* Examples of this for $\pi N \leftrightarrow \pi N$, $\pi\pi \rightarrow N\bar{N}$ and $NN \rightarrow NN$ scattering are given in ref. (23) and for $\gamma N \leftrightarrow \gamma N$ in ref. (24).

and fixed t for the amplitudes $h^{(t)}$, and, depending on the degree of convergence they have at high s, we may even write superconvergence relations like (45). To this end, we find from the limit

$$\cos\vartheta \xrightarrow[s\to\infty]{} 2st\left[(t-(m_1+m_3)^2)(t-(m_1-m_3)^2)(t-(m_2+m_4)^2)(t-(m_1-m_4)^2)\right]^{-1/2} \tag{48}$$

that

$$\left| h^{(t)}_{\lambda_4\lambda_2\lambda_3\lambda_1} \right| \xrightarrow[s\to\infty]{} c(t) \left| H^{(t)}_{\lambda_4\lambda_2\lambda_3\lambda_1} \right| / s^{n(\nu,\mu)} \tag{49}$$

where $n(\nu, \mu)$ equals the maximum of ν and μ, and $c(t)$ may be determined from (48). It is clear from (49) that the larger the helicity flip involved in $h\left\{\begin{smallmatrix} t \\ \lambda \end{smallmatrix}\right\}$, the better its convergence will be at large z. It remains therefore to determine the bounds on $H^{(t)}$ to know the behaviour of $h^{(t)}$ at large a. This can be obtained by use of the crossing relations from the bounds on $H^{(s)}$ which we assume are known[21] independently.

It is at this point that the analysis differs between processes which include massless particles and those which do not. The difference comes through the different asymptotic properties of the analytically continued little group elements corresponding to each particle in the crossing relations (5).

In the massive case (5) takes the form (22) and because of the reality of the angle $\bar{\omega}$, c.f. (27), $d^{(j)}(\bar{\omega})$ does not contribute to the asymptotic s behaviour of $H^{(t)}$ in addition to the contribution coming from the bound on $H^{(s)}$, $H^{(s)}$ $\zeta(s, t)$ say. Then the bound on $h^{(t)}$ will be

$$\left| h^{(t)}_{(\lambda)}(s,t) \right| < c(t)\, \xi(s,t)\, s^{-n(\nu,\mu)} \tag{50}$$

For example, the amplitude $h_{1-1;00}$ in $\pi\rho$ scattering has the asymptotic behaviour $\zeta(s, t)s^{-2}$.

Applying the same procedure to a process involving a massless particle of spin j we have to take into account the asymptotic contribution of the factor in the crossing relation corresponding to this particle.

Here we are interested in the inverse of the element given by (37) for we are crossing from the s- to the t-channel. Thus we can see by inspection of (37) and (37') that for every massless spin j particle in(out)going in the t-channel, there will be a factor of $s^{\mp\frac{1}{2}j} = (s^{\mp\frac{1}{2}})^j = e^{ij\overline{\phi}}$. The analog of (50), the s-bound on $h^{(t)}$ then becomes

$$\left| h^{(t)}_{(\lambda)}(s,t) \right| < c(t)\, g(t)\, \xi(s,t)\, s^{-n(\nu,\mu)-\sum_n j_n+\sum_m j_m} \tag{51}$$

where j_n j_m are the spins of the massless particles in(out)going massless particles in the t-channel; and $g(t)$ can be determined from (37). For the $\pi\rho$ amplitude $h^{(t)}_{1,-1,0,0}$ mentioned above, the convergence is now $\zeta(s,t)s^{-2\pm1}$.

CHAPTER VI

ON EQUAL TIME COMMUTATORS OF SL(2, C) CURRENTS

In its simplest form, the method of Current Algebra[25] consists of taking matrix elements of equal-time current commutators which form an algebra* and then saturating these commutators with intermediate states, with consideration for the quantum numbers of the particles - between states of which the matrix elements of these currents are taken - with respect to the quantum numbers of the currents themselves. Relations may be obtained thereby, between the form factors of these currents and under some energetic restrictions, coupling constants may be compared which can be checked against experiment.

These currents have definite transformation properties under H.L.G. and some internal symmetry (compact unitary) Group. In generating the algebrae of the currents, their internal symmetry properties act completely independently of their H.L.G. properties, so we shall ignore them completely in the following paragraphs, since we shall be interested only in the space-time properties of currents and their algebrae.

In generating the algebrae of currents, it is customary to take the current to be a bilinear in spin ½ quark fields[25], and by using the equal-time anti-commutation properties (III.27) for the spin ½ fields, the Algebra of the currents is generated. Such currents are given by

---

* The equal-time commutator of two currents gives back a current already in the Algebra, in general after several commutations.

$$J_{(\mu),m}(x) = \overline{\psi}(x)\ \Lambda_m^{(M)} \otimes \Gamma_{(\mu)}\,\psi(x) \tag{1}$$

where the quark fields $\psi(x)$ are the spherical basis[25] of the representation (M) - $M \equiv$ multiplicity - of the internal symmetry group, and **independently** are the basis for the $(\frac{1}{2},0) + (0,\frac{1}{2})$ representation of H.L.G. The matrix $\Lambda_m^{(M)}$ is the m-th generator of the internal group in the spherical basis and since the quarks belong to the basic representations (3) and (2) for the SU(3) and SU(2) examples respectively, the only possible representations the current can have under these groups, follow from the Clebsch Gordan (C.G.) series to be (1) or (8) and (1) or (3) respectively. The matrices $_{(\mu)}$, which are more interesting to us, are combinations of Dirac γ-matrices, with $(\mu) = (\mu_1, \mu_2, \ldots, \mu_n)$ for an n-th order tensor current $(n = 0,1,2\ldots)$. The most commonly investigated currents hitherto have been the scalar, pseudo-scalar (spin 0); vector, axial vector (spin 1) currents, although second order tensor[26] (spin 2) currents, and even spin $\frac{1}{2}$ currents[27] have been considered.

The current given in (1) transforms by the Lorentz transformation matrices $\Lambda_\mu{}^\nu$ and not by the SL(2, C) (c.f. Section 1 paragraph 4) matrices $A[\Lambda]$. It is our purpose in this section to construct and then investigate the possibility of using SL(2, C) currents. We shall be restricting our attention almost entirely to currents transforming according to the (1, 0) representation of H.L.G., and, will not even in principle consider $(j/2, 0)$ type currents so that we remain entirely within the quark model, that is, work with currents defined as bilinears in quarks, although, had we an extension of the quark-model to integral-spin quarks, the procedure described in this section would be quite adequate to treat the half-integral spin currents on the same footing as integral spin currents.

We believe that the SL(2, C) approach has two main advantages over the O(3, 1) approach, both of which essentially stem from the feature of the SL(2, C) representations that the (j, 0) representation of H.L.G. is (2j + 1)-dimensional and hence has exactly the same multiplicity as the spin states of a spin-j particle. On the other hand, it follows from (II.18) that the operation of the representation matrices $\Lambda^{(\mu)}{}_{(\nu)}$ of O(3, 1) onto a symmetric tensor basis correspond to the $({}^{j}/2, {}^{j}/2)$ representation of H.L.G. for a particle of spin-j. Now $({}^{j}/2, {}^{j}/2) = ({}^{j}/2, 0) \otimes (0, {}^{j}/2)$ is not an irreducible representation and in fact corresponds to particles of spins j, j-1,..., 0 through the C.G. series. For example, A_μ representing a spin-1 particle has four components, and to reduce this to the actual number of three projections for spin-1, one has to use one condition, the so-called Lorentz condition $p^\mu A_\mu = 0$. If we used A_μ for a spin-2 particle, it follows from the C.G. series above that one has to use four conditions - and so on.

The first advantage of this formalism is the expression of form factors for arbitrary spins by a simple prescription. To write down the form factors of a vertex, one has to construct bilinears of fields - which are themselves expansions of the creation and annihilation operators of the two particles between states of which the matrix elements of the current is taken - such that they can couple to an integral spin particle with exactly the same quantum numbers as the current in the original Algebra. These couplings are in general multiple, and occur through different orbital angular momenta, so that these "currents" formed of two fields also include the relativistic solid harmonics. In fact this is exactly what we have done in writing down vertex functions for arbitrary spin in paragraph 3 of Section III, and the general result (III.46) may be applied directly to any case in the present context simply by identifying j_3 with the transformation $(j_3, 0)$ of the $SL(2, C)$ current in the Algebra.

The second advantage occurs in the analysis of deriving dispersion relations by making use of equal-time current commutators[28]. Dispersion relations can be written for the invariant amplitudes given in the expansion of the M-function with respect to some Lorentz covariant basis functions (c.f. Section II). In the above analysis, the M-function for a four particle process is written in terms of its reduction[28] with respect to two three-particle functions, which come in as a current commutator. Now if these currents transform with the conventional $(\frac{1}{2}, \frac{1}{2})$ representations of H.L.G., that is with $\Lambda^{\mu}{}_{\nu}$, then the M-functions will be labelled by indices (μ). This labelling will result in a larger number of scattering functions than there would be in the corresponding scattering process of spinning particles. It is exactly this redundancy that will be avoided by using the

(1, 0)-type irreducible currents, and the four-particle functions resulting will be SL(2, C) M-functions, whose decomposition into invariant amplitudes has been analysed by several authors( ).

We shall construct (1, 0) currents and calculate their equal-time commutators in paragraph 2. In paragraph 3 we shall try to use these current commutators to derive dispersion relation sum rules.

2. Equal-time Commutation Relation for $J_\alpha^{(1,\,0)}$:

The definition of $J_\alpha^{(1,\,0)}$ current as a bilinear of the quark fields $\phi_\sigma^{(\frac{1}{2},0)}$ and $\chi_\sigma^{(0,\frac{1}{2})}$ is

$$J^{(1)}_\alpha(x) = [\tfrac{1}{2}\tfrac{1}{2}1]_\alpha^{\sigma_1\sigma_2}\,[C^{-1}\chi^\dagger(x)]_{\sigma_2}\,\varphi_{\sigma_1}(x)$$

$$= (-)^{\frac{3}{2}+\sigma_2}\,[\tfrac{1}{2}\tfrac{1}{2}1]_\alpha^{\sigma_1\sigma_2}\,\chi^\dagger_{-\sigma_2}(x)\,\varphi_{\sigma_1}(x) \tag{2}$$

whence we can define also the (0, 1) current $\bar{J}^{(1)}$ by using (I.99)

$$\bar{J}^{(1)}_\alpha(x) = C^{-1}_{\alpha\alpha'}\,J^{(1)\dagger}_{\alpha'}(x)$$

$$= (-)^{\frac{1}{2}+\alpha+\sigma_2}\,\{\tfrac{1}{2}\tfrac{1}{2}1\}_{-\alpha}^{\sigma_1\sigma_2}\,\varphi^\dagger_{\sigma_1}(x)\,\chi_{-\sigma_2}(x)\,. \tag{3}$$

It is clear that if we are interested in constructing higher order currents, $J^{(n)}$ $(n = 2,3,\ldots)$, out of spin $\frac{1}{2}$ quarks, we shall have to include the orbital angular momentum to the coupling (c.f. Sec. III, para. 3)

$$J_\alpha^{(j)}(x) = \left[\tfrac{1}{2}\,\tfrac{1}{2}\,s\right]_\sigma^{\sigma_1\sigma_2}\left[s\,\ell\,j\right]_\alpha^{\sigma m}\left[C^{-1}\chi^\dagger(x)\right]_{\sigma_2}\varphi_{\sigma_1}(x)\,\mathcal{Y}_m^{(\ell)}(\partial^{(1)},\partial^{(2)}) \tag{4}$$

The difficulty in calculating equal-time commutators of such currents is, that we are faced with equal-time commutators of fields with derivatives of fields like $\partial_\mu \phi(x)$, which are not considered in paragraph 2 of Section III. Naturally, we may try to use the equation of motion (Dirac equation)

$$\begin{aligned} -i\tilde{\sigma}^\mu \partial_\mu \varphi(x) &= m\,\chi(x) \\ -i\sigma^\mu \partial_\mu \chi(x) &= m\,\varphi(x) \end{aligned} \tag{5}$$

to enable us to write, for example, $\left[\partial_0 \phi(x),\ \phi^+(y)\right]_{x_0=y_0}$ in terms of $\left[\partial_i\phi(x),\ \phi^+(y)\right]_{x_0=y_0} = \partial_i\left[\phi(x),\ \phi^+(y)\right]_{x_0=y_0}$ which is in fact equal to a derivative $\delta^{(3)}$-function. Unfortunately apart from the fact that a derivative $\delta^{(3)}$-function is not useful in reproducing currents, it is not in the first place correct to use (5), because these hold only for free quark fields. It appears that the problem of accounting for equal-time commutators of currents with derivatives of currents is tied up with our knowledge of the interaction of quark fields.

Alternatively, it might seem to one that derivatives such as those appearing in (4) could be avoided by using a more artificial "quark" model, with quarks of spin-$\frac{1}{2}$, such that

$$J^{(1)}_{\alpha}(x) = [\tfrac{1}{2}\,\tfrac{1}{2}\,1]^{\sigma_1\sigma_2}_{\alpha}\,[C^{-1}\,\chi^{(\frac{1}{2})\dagger}(x)]_{\sigma_2}\,\varphi^{(\frac{1}{2})}_{\sigma_1}(x) \tag{6}$$

This, however, is not the solution to our problem, because in view of our considerations in paragraph 2 of Section III, we know that equal-time commutators of fields with higher spin than one half include derivative $\delta^{(3)}$-functions, which are not useful in reproducing currents, that is, an Algebra.

It is therefore clear that to generate algebrae for currents of higher order than one, we would need to make a dynamical assumption to account for equal-time commutators of currents with derivatives of currents. In other words, we would be faced by our ignorance of interacting quark fields.

In the following, we shall restrict our interest solely to the (1, 0) currents, but of course everything that we do can be repeated more easily for (0, 0) currents, which emerge directly from (2) if we replaced the C.G. coefficient $[\frac{1}{2}\,\frac{1}{2}\,1]^{\sigma_1\sigma_2}_{\alpha}$ by $[\frac{1}{2}\,\frac{1}{2}\,0]^{\sigma_1\sigma_2}_{\alpha}$.

To evaluate the equal-time commutator of $J^{(1)}_{\alpha}$, we start by presenting two simple formal tools: (a) the connection between C.G. coefficients and Pauli spin matrices, (b) the connection between the Cartesian basis and the spherical basis.

(a) C.G. Coefficients and Pauli matrices:

Consider the two-component spinors ξ_{σ_1} and ξ_{σ_2} to form a 3-component-spinor (spherical-vector) by

$$[\tfrac{1}{2}\tfrac{1}{2}1]_{\alpha}^{\sigma_1\sigma_1} \xi_{\sigma_2} \xi_{\sigma_1} = \xi^{\sigma_2} (\sigma_{\alpha}^{(1)})_{\sigma_2\dot{\sigma}_1} \xi^{\dot{\sigma}_1} \tag{7}$$

where $\sigma^{(1)}$ is a Pauli matrix in the spherical basis. Since these spinors transform under SU(2, C) matrices which are unitary, it follows from our remarks about (68), and by using (69), that

$$[\tfrac{1}{2}\tfrac{1}{2}1]_{\alpha}^{\sigma_1\sigma_1} \xi_{\sigma_2} \xi_{\sigma_1} = (-)^{\frac{1}{2}-\sigma_2} (\sigma_{\alpha}^{(1)})_{-\sigma_2\dot{\sigma}_1} \xi_{\sigma_2} \xi_{\sigma_1} . \tag{8}$$

Since this is true for arbitrary and , it follows that

$$[\tfrac{1}{2}\tfrac{1}{2}1]_{\alpha}^{\sigma_2\sigma_1} = (-)^{\frac{1}{2}-\sigma_2} (\sigma_{\alpha}^{(1)})_{-\sigma_2\dot{\sigma}_1} . \tag{9}$$

Similarly, using the C.G. for the complex conjugate representation

$$\{\tfrac{1}{2}\tfrac{1}{2}1\}_{\dot{\alpha}}^{\dot{\sigma}_1\dot{\sigma}_2} \xi_{\dot{\sigma}_1} \xi_{\dot{\sigma}_2} = \xi^{\sigma_1} (\sigma_{\dot{\alpha}}^{(1)})_{\sigma_1\dot{\sigma}_2} \xi^{\dot{\sigma}_2} \tag{10}$$

we find that

$$\{\tfrac{1}{2}\tfrac{1}{2}1\}_{\dot{\alpha}}^{\dot{\sigma}_1\dot{\sigma}_2} = (-)^{\frac{1}{2}-\dot{\sigma}_2} (\sigma_{\dot{\alpha}}^{(1)})_{\sigma_1, -\sigma_2} . \tag{11}$$

(b) The relation between Cartesian and spherical bases:

Define e_{α}^{i} by

$$J_{\alpha}^{(1)} = e_{\alpha}^{\ i} J_i \tag{12}$$

where J_i and $J^{(1)}_\alpha$ are Cartesian ($i = 1,2,3$) and Sperical ($\alpha = 1,0,-1$) tensor operators( ) of order one, with

$$\vec{J} \cdot \vec{J} = (-)^\alpha J_\alpha J_{-\alpha} \tag{13}$$

These J_i can be the generators of SU(2, C).

Taking the complex conjugate of e^i_α is denoted, according to (68), by dotting the 3-component spinor index α, whence, using (69) as in (a) above, we have

$$e_\alpha{}^{i*} = (-)^\alpha e_{-\alpha}{}^i \tag{14}$$

Substituting (12) into (13) and using (14), we get

$$e_\alpha{}^i e_\alpha{}^{j*} = \delta_{ij} \tag{15}$$

which is the definition of the inverse of $e_\alpha{}^i$.

Making use of (15) in (12) and then substituting (12) into (13) we have

$$\begin{aligned} \vec{e}_\alpha \cdot \vec{e}_\beta^{\,*} &= \delta_{\alpha\beta} \\ \vec{e}_\alpha \cdot \vec{e}_\beta &= -[1\,1\,0]^{\alpha\beta}_{0}. \end{aligned} \tag{16}$$

Finally, consider

$$J^{(1)}_\gamma = [1\,1\,1]^{\alpha\beta}_\gamma \, J^{(1)}_\alpha J^{(1)}_\beta \tag{17}$$

and

$$J_i J_j = \frac{1}{4} \delta_{ij} + \frac{1}{2} \varepsilon_{ijk} J_k \tag{18}$$

which follows from the properties of tensor operators and where ε_{ijk} is the totally antisymmetric tensor.

Substituting first (12) and then (18), in (17), we find by further using the identity

$$\varepsilon_{\ell m k}\, \varepsilon_{ijk} = \delta_{\ell i}\, \delta_{mj} - \delta_{\ell j}\, \delta_{mi} \tag{19}$$

that

$$\varepsilon_{ijk}\, e_\alpha{}^i\, e_\beta{}^j\, e_\gamma{}^{k*} = -i\, [111]_\gamma^{\alpha\beta} . \tag{20}$$

We are now in a position to generate the algebra of currents, which includes the (1, 0) currents.

The equal time commutator between $J_\alpha(x)$ in (2) and its hermitian conjugate at another space point $y(y_o = x_o)$

$$J^\dagger_{\dot\beta}(y) = (-)^{3/2 + \dot\sigma_2'} \left\{ \tfrac{1}{2}\ \tfrac{1}{2}\ 1 \right\}^{\dot\sigma_1' \dot\sigma_2'}_{\dot\beta'} \varphi^\dagger_{\dot\sigma_1'}(y)\, \chi_{-\dot\sigma_2'}(y) \tag{21}$$

reduces, by the use of (III.27) and the corresponding result for χ, which is the same, to

$$[J_\alpha(x), J^\dagger_\beta(y)]_{x_0=y_0} = (-)^{1+\sigma_1+\sigma_1'} [\tfrac{1}{2}\tfrac{1}{2}1]^{\sigma_1\sigma_2}_{\alpha} \{\tfrac{1}{2}\tfrac{1}{2}1\}^{\sigma_1'\sigma_2'}_{\beta} \delta(\vec{x}-\vec{y})$$
$$[\chi^\dagger_{-\sigma_2}(x)\, \delta_{\sigma_1\sigma_1'}\, \chi_{-\sigma_2'}(y) - \varphi^\dagger_{\sigma_1'}(y)\, \delta_{\sigma_2\sigma_2'}\, \varphi_{\sigma_1}(x)] \tag{22}$$

Substituting (9) and (11) into (22) for the C.G. coefficients, we get

$$[J_\alpha(x), J^\dagger_\beta(y)]_0 = \delta(\vec{x}-\vec{y})[\chi^\dagger(x)\, \sigma^{(1)}_\alpha \sigma^{(1)}_\beta\, \chi(y) - \varphi^\dagger(y)\, \sigma^{(1)}_\beta \sigma^{(1)}_\alpha\, \varphi(x)] \tag{23}$$

in matrix notation. Remembering that $\vec{J}^{(1/2)} = \tfrac{1}{2}\vec{\sigma}$, substituting (12) into (23) and then using (18), (16) and (20) we arrive at

$$[J_\alpha(x), J^\dagger_\beta(y)]_0 = \delta_{\alpha\beta}\, \delta(\vec{x}-\vec{y})[\chi^\dagger(x)\chi(y) - \varphi^\dagger(y)\varphi(x)]$$
$$+ [111]^{\alpha\beta}_\gamma\, e_\gamma{}^k\, \delta(\vec{x}-\vec{y})\, [\chi^\dagger(x)\sigma_k\chi(y) + \varphi^\dagger(y)\sigma_k\varphi(x)] \tag{23'}$$

and using the notation given by (III.51) we have

$$[J_\alpha(x), J^\dagger_\beta(y)]_0 = i\,\delta_{\alpha\beta}\, J^V_0(x) + i\,[111]^{\alpha\beta}_\gamma\, e_\gamma{}^k\, J^V_k(x) \tag{24}$$

after formally having carried out the δ-function integration, and, with

$$J^A_\mu(x) = i\,\bar{\psi}(x)\,\gamma_\mu\gamma_5\,\psi(x) \tag{25}$$

the familiar axial vector current. It is not surprising that by commuting two spin one currents we end up with an axial vector current, because the result of space inversion on $J(x_o, \vec{x})$, as predicted by (III.16) is $J^+(x_o, -\vec{x})$ up to a similarity transformation, and vice versa, so that their antisymmetric combination (commutator) should have odd symmetry under space inversion.

The commutator in (24) does not form an algebra. The equal-time commutator of J_α and J^+_β generates two new elements $Q^V = J^V_o = e_\gamma{}^k J^V_k$. We close the algebra by evaluating the equal-time commutators:

$$[Q^V(x), J_\alpha(y)]_o = 2J_\alpha(x)$$

$$[J^V_\alpha(x), J_\beta(y)]_o = 2[111]^{\alpha\beta}_\gamma J_\gamma(x). \qquad (26)$$

(24) and (26) together give the algebra of J, J^+, Q^V and J^V, and subsequently the internal symmetry group may also be incorporated, then matrix elements between two one-particle states may be taken, and after saturation of the intermediate states relations between form factors can be obtained.

We shall not be concerned with this practical application here. Instead, the only result we shall use is (24) which will be useful in the next paragraph for the derivation of dispersion relations from equal-time current commutators.

3. Equal-time Commutators and Dispersion Relations

We start this paragraph by briefly presenting the analysis[28] in space-time, and then we go on with our attempt to carry it through in the three-dimensional complex space.

Define

$$\text{(a)} \quad T_{\mu\nu}[K] = i\int d^4x\, \theta(x_0)\, e^{ikx} \langle p_1 | [J_\mu(x), J_\nu(0)] | p_2 \rangle$$

$$\text{(b)} \quad t_{\mu\nu}[K] = \int d^4x \qquad e^{ikx} \langle p_1 | [J_\mu(x), J_\nu(0)] | p_2 \rangle \tag{27}$$

so that $T_{\mu\nu}$ is the Hilbert transform of $t_{\mu\nu}$, with p_1 and p_2 the momenta of spinless particles for simplicity, and $[K]$ stands for $[k, q; p_1, p_2]$ the four fourmomenta in the scattering process. Generalising from our experience in local field perturbation theory we assume that $T_{\mu\nu}$ and $t_{\mu\nu}$ can be decomposed into scalar functions, which themselves are functions of the scalars formed from $[K]$, with respect to some covariant bases $Y^{(i)}_{\mu\nu}$. This is the expansion into invariant amplitudes that we mentioned in Section II. Let us suppose that we have such a decomposition on hand:

$$T_{\mu\nu}[K] = \sum_i^n \mathcal{A}^{(i)}(s,t,u)\, Y^{(i)}_{\mu\nu}[K]$$

$$t_{\mu\nu}[K] = \sum_i^n a^{(i)}(s,t,u)\, Y^{(i)}_{\mu\nu}[K] \tag{28}$$

where s, t, u are the scalars formed from $[K]$, and n is the number of independent invariant amplitudes, ten in this case, although four of them should be redundant because we know that there are only six independent spin-amplitudes in this case, with positive parity signature.

To make use of equal-time commutators, we need to insert a differential operator in the integrand of (27a), whose operation on $\theta(x_o)$ gives $\delta(x_o)$ and hence an equal-time commutator. This can be done by multiplying both sides of (27a) by k_μ, and thus taking k_μ into the integrand on the right-hand side. The result of doing this is

$$\begin{aligned} k_\mu T_{\mu\nu}[K] &= U_\nu[K] + 2G(t)(p_1+p_2)_\nu \\ k_\mu t_{\mu\nu}[K] &= u_\nu[K] \end{aligned} \tag{29}$$

where

$$\begin{aligned} U_\nu[K] &= i\int d^4x\ \theta(x_o)\, e^{ikx} \langle p_1 | [\partial_\mu J_\mu(x), J_\nu(0)] | p_2 \rangle \\ u_\nu[K] &= \int d^4x\ \ e^{ikx} \langle p_1 | [\partial_\mu J_\mu(x), J_\nu(0)] | p_2 \rangle \end{aligned} \tag{30}$$

and

$$\langle p_1 | [J_o(x), J_\nu(0)]_{x_o=0} | p_2 \rangle = 2G(t)(p_1+p_2)_\nu\, \delta(\vec{x}) \tag{31}$$

U [K] being the Hilbert transform of u [K] . Similarly to (28) we have

$$\begin{aligned} U_\nu[K] &= \sum_{i=1}^{3} B^{(i)}(s,t,u)\, Y_\nu^{(i)}[K] \\ u_\nu[K] &= \sum_{i=1}^{3} b^{(i)}(s,t,u)\, Y_\nu^{(i)}[K]. \end{aligned} \tag{32}$$

It follows from (28) and (32) that we can write down the following dispersion relations

$$\mathcal{A}^{(i)}(s) = \frac{1}{\pi}\int \frac{a^{(i)}(s')}{s'-s}\, ds'$$
$$\mathcal{B}^{(i)}(s) = \frac{1}{\pi}\int \frac{b^{(i)}(s')}{s'-s}\, ds'. \qquad (33)$$

Substituting (28) and (32) into (29) and then comparing coefficients of $(p_1 + p_2)_\mu$ and using (33) one obtains

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{Im}\, \mathcal{A}^{(i)}(s,t,u)\, ds = 2G(t). \qquad (34)$$

This is the type of relation we set out to derive from dispersion relations and equal-time commutators.

Following this procedure, our object is now to set up the analogous relations to those given by (27) to (33). If we can do this, then the problem of deriving relations like (34) is, at least in principle, solved.

To this end we define

$$M_{\alpha\dot\beta}[K] = i\int d^4x\ \theta(x_0)\ e^{ik\cdot x} \langle p_1|[J_\alpha(x), J_{\dot\beta}^{\dagger}(0)]|p_2\rangle$$
$$m_{\alpha\dot\beta}[K] = \int d^4x\ \theta(x_0)\ e^{ikx} \langle p_1|[J_\alpha(x), J_{\dot\beta}^{\dagger}(0)]|p_2\rangle \qquad (35)$$

where now the functions M and m have the transformation properties of SL(2, C) M-functions and let us suppose their decompositions into invariant amplitudes[6,29] is given by

$$M_{\alpha\dot\beta}[K] = \sum_i^4 A^{(i)}(s,t,u)\, Y^{(i)}_{\alpha\dot\beta}[K]$$
$$m_{\alpha\dot\beta}[K] = \sum_i^4 a^{(i)}(s,t,u)\, Y^{(i)}_{\alpha\dot\beta}[K]. \qquad (36)$$

Now the problem of multiplication with momenta, to find the analogues of (29) is somewhat more complicated in this formalism. Clearly, to preserve the covariance of the theory, these factors of momenta will have to come in covariant from, that is as spinors $k(\mu)\,\tilde{\rho}^{(\mu)}(j, j')$ transforming according to the (j, j') representation of H.L.G. in general. In particular in this case, $M[K]$ and $m[K]$ both transform according to the $(1, 1)$ representation, and according to our remarks in Section II, their spinor bases $Y^{(i)}_{\alpha\beta}\,K$ are of the form $\tilde{\rho}^{(\mu)}(1, 1)\,k_{(\mu)} = t^{\mu\nu}\,k_\mu\,k_\nu$, so that if we expect to have any simplification at all when making the substitution analogous to (28) into (29), we must define the analogoues of (29) as $D^{(1)}[k.\tilde{\sigma}]\,M[K] = \bar{t}^{\mu\nu}\,k_\mu\,k_\nu\,M[K]$ and $D^{(1)}[k.\tilde{\sigma}]\,D^{(1)}[k.\sigma] = m^4\,1$, where $k^2 = m^2$. Proceeding as for (29) we find

$$
\begin{aligned}
D^{(1)}_{\alpha\beta}[k.\tilde{\sigma}]\,M_{\beta\gamma}[K] &= -\mathcal{U}_{\alpha\gamma}[K] - k_\mu\,\bar{t}^{\mu o}_{\alpha\beta}\,F_{\beta\gamma}[K] - X_{\alpha\gamma}[K] \\
D^{(1)}_{\alpha\beta}[k.\tilde{\sigma}]\,m_{\beta\gamma}[K] &= -u_{\alpha\gamma}[K]
\end{aligned}
\tag{37}
$$

where

$$
\begin{aligned}
\mathcal{U}_{\alpha\gamma}[K] &= \bar{t}^{\mu\nu}_{\alpha\beta}\int d^4x\;\theta(x_o)\,e^{ikx}\langle p_1|\,[\partial_\mu\partial_\nu J_\beta(x),\,J^\dagger_\gamma(0)]\,|p_2\rangle \\
u_{\alpha\gamma}[K] &= \bar{t}^{\mu\nu}_{\alpha\beta}\int d^4x\quad e^{ikx}\langle p_1|\,[\partial_\mu\partial_\nu J_\beta(x),\,J^\dagger_\gamma(0)]\,|p_2\rangle
\end{aligned}
\tag{38}
$$

$$
\begin{aligned}
\mathcal{U}_{\alpha\gamma}[K] &= \sum_i^4 B^{(i)}(s,t,u)\,Y^{(i)}_{\alpha\gamma}[K] \\
u_{\alpha\gamma}[K] &= \sum_i^4 b^{(i)}(s,t,u)\,Y^{(i)}_{\alpha\gamma}[K]
\end{aligned}
\tag{39}
$$

and
$$
F_{\beta\gamma}[K] = \langle p_1|\,[J_\beta(x),\,J^\dagger_\gamma(0)]_{x_o=0}\,|p_2\rangle \tag{40}
$$

which can be determined from (24). Here we have introduced the analogues of all relations (27) to (32) and relations similar to (33) in A and B also hold. The problem would be solved if we could determine $X_{\alpha\gamma}$ in the first member of (37), which proves to be the stumbling block:

$$X_{\alpha\gamma}[K] = i\,\bar{t}^{\,0\nu} \int d^4x\, \delta(x_0)\, e^{ikx} \langle p_1 | [\partial_\mu J_\alpha(x), J_\gamma^\dagger(0)] | p_2 \rangle \qquad (41)$$

It involves the equal-time commutator of a current with the derivative of a current. As we mentioned in the first paragraph of this section, this kind of object cannot be evaluated unless we knew the equation of motion for an interacting quark field, or alternatively, if we had a physical hypothesis on hand, of the same nature as P.C.A.C.[30]. Using the free field equations in the hope that one might be able to guess what the interaction term would be, and trying to determine (41), turned out to be extremely tedious and unprofitable, because one was left with integrands including derivative $\delta^{(3)}$-functions which were not useful in reproducing a bilinear which could be regarded as another current, and because the mass of the quark appeared explicitly.

It is not surprising that we encounter this difficulty, because we have got two factors of momentum in $D^{(1)}[k.\sigma]$. We could of course have used $(k.o)$ alone in (37), but this would have complicated immensely the transformation properties of the theory, under H.L.G.

In a particular case, however, we can avoid this difficulty relatively simply. This is the case when one of the momenta, corresponding to a spin 1 index in $M[K]$ has vanishing norm $k^2 = 0$, that is it corresponds to a massless particle. The crucial point is the theorem(7) on M-functions for massless particles, given in paragraph 4 of Section II, which states that, if the M-functions describe a process with an in or out-coming massless particle of spin j and momentum k_1, then

$$M[K]\, D^{(j)}[k_1 \cdot \tilde{\sigma}] = 0$$
$$D^{(j)}[k_1 \cdot \tilde{\sigma}]\, M[K] = 0 \tag{42}$$

If now the other spin 1 particle with momentum k_3 is massive then $D^{(1)}[k_3 \cdot \tilde{\sigma}]\, M[K]$ does not vanish, and we can substitute for the $\mathfrak{X}$ - term in it from the identity $M[K]\, D^{(1)}[k_1 \cdot \sigma] = 0$.

To illustrate this we define a slightly modified version of the previous M-functions in (35):

$$M_{\alpha\beta}[K] = i \iint d^4x\, d^4y\, \theta(x_0 - y_0)\, e^{i(k_1 x - k_3 y)} \langle k_2 | [J_\alpha(x), J_\beta^\dagger(y)] | k_4 \rangle \tag{43}$$

where $[K] = [k_4, k_3, k_2, k_1]$ and k_2, k_4 are still momenta of spinless particles for simplicity. A similar definition is made for $m_{\alpha\beta}[K]$. Since our only interest here is to deal with $\mathfrak{X}$, we shall only deal with the $M[K]$ in the following.

Proceeding as before, we find that

$$D^{(1)}_{\gamma\beta}[k_3 \cdot \tilde{\sigma}]\, M_{\beta\alpha}[K] = -k_\mu\, \bar{t}^{\mu o}_{\gamma\beta}\, F_{\beta\alpha}[K] - \mathfrak{U}_{\gamma\alpha}[K] + \mathfrak{X}^{(3)}_{\gamma\alpha}[K] \tag{44}$$

where

$$\mathcal{U}_{\gamma\alpha}[K] = i\, t^{\mu\nu}_{\gamma\beta} \iint d^4x\, d^4y\, \theta(x_0 - y_0)\, e^{i(k_1 x - k_3 y)} \langle k_2 | [J_\beta(x), \partial_\mu \partial_\nu J^\dagger_\alpha(y)] | k_4 \rangle \quad (45)$$

and

$$\mathfrak{X}^{(3)}_{\gamma\alpha}[K] = i\, t^{0\nu}_{\gamma\beta} \iint d^4x\, d^4y\, \delta(x_0 - y_0)\, e^{i(k_1 x - k_3 y)} \langle k_2 | [J_\beta(x), \partial_\nu J^\dagger_\alpha(y)] | k_4 \rangle \quad (46)$$

For k_1 on the other hand, we get, by using (42a)

$$\mathfrak{X}^{(1)}_{\alpha\gamma}[K] = \mathcal{V}_{\alpha\gamma}[K] + F_{\alpha\beta}[K]\, \bar{t}^{\mu 0}_{\beta\gamma}\, k_\mu \quad (47)$$

where

$$\mathcal{V}_{\alpha\gamma}[K] = i \iint d^4x\, d^4y\, \theta(x_0 - y_0)\, e^{i(k_1 x - k_3 y)} \langle k_2 | [\partial_\mu \partial_\nu J_\alpha(x), J^\dagger_\beta(y)] | k_4 \rangle\, \bar{t}^{\mu\nu}_{\beta\gamma} \quad (48)$$

and

$$\mathfrak{X}^{(1)}_{\alpha\gamma}[K] = i \iint d^4x\, d^4y\, \delta(x_0 - y_0)\, e^{i(k_1 x - k_2 y)} \langle k_3 | [\partial_\nu J_\alpha(x), J^\dagger_\beta(y)] | k_4 \rangle\, t^{0\nu}_{\beta\gamma} \,. \quad (49)$$

We can easily see now that

$$\begin{aligned} \mathfrak{X}^{(3)}[K] &= \mathfrak{X}^{(1)\dagger}[K']\,; \\ [K] &= [k_4, k_3, k_2, k_1] \\ [K'] &= [k_2, k_1, k_4, k_3]\,, \end{aligned} \quad (50)$$

so that, using (47) and (50), we can replace $X^{(3)}$ in (44), in terms of $\mathcal{V}[K]$ and $F[K]$, thus rendering the rest of the analysis feasible.

REFERENCES

Bibliography for:

Section I : Refs. (1), (2), (3).

Section II : Refs. (6), (7), (8).

(1) E.P. Wigner, Ann. Math., 40, 149 (1939).

(2) S. Weinberg, Brandeis Notes (1964), Vol. 2.

(3) P. Moussa and R. Stora, Hercegnovi Notes, Dept. 1966.

(4) A. Messiah, Quantum Mechanics, North Holland Publ. Co., 1961.

(5) H.P. Stapp, Phys. Rev., 125, 2139 (1962).

(6) A.O. Barut, I. Muzinich, D.N. Williams, Phys. Rev. 130, 442 (1963).

(7) D. Zwanziger, Phys. Rev. 133, B1036 (1964).

(8) A.O. Barut, The Theory of the Scattering Matrix, Macmillan & Co., New York, 1967.

(9) M. Jacob and G.C. Wick, Ann. Phys. (N.Y.) 7, 404 (1959).

(10) S. Weinberg, Phys. Rev. 133, B1318 (1964); Phys. Rev. 134, B882, (1964).

(11) H. Joos, Fortschritte der Physik 10, 65 (1962).

(12) (a) F. Zachariasen and C. Zemach, Phys. Rev. 128, 849 (1962)
(b) E. Abers and C. Zemach, Phys. Rev. 131, 2305 (1963).

(13) G.F. Chew, S-matrix Theory of Strong Interactions, W.A. Benjamin, Inc., New York, 1962.

(14) S. Schweber, Relativistic Quantum Field Theory, Row, Peterson & Co., Elmsford, New York, 1961.

(15) K. Nishijima, Fundamental Particles, W.A. Benjamin Inc., New York, 1963.

(16) M.S. Marinov and V.I. Roginsky, Nucl. Phys. 49, 251 (1963).

(17) T.L. Trueman and G.C. Wick, Ann. Phys. (N.Y.) 26, 332 (1964).

(18) I. Muzinich, J. Math. Phys., 5, 1481 (1964).

REFERENCES (Contd.)

(19) T.W.B. Kibble, Phys. Rev. 117, 1159 (1960).

(20) V. de Alfaro, S. Fubini, G. Rossetti and G. Furlan, Phys. Letters 21, 576 (1966).

(21) T.L. Trueman, Phys. Rev. Letters 17, 1198 (1966).

(22) M. Gell-Mann, M.L. Goldberger, F.E. Low, E. Marx and F. Zachariasen, Phys. Rev. 136, B507 (1964; F. Calogero, J.M. Charap and E.J. Squires, Ann. Phys. (N.Y.) 25, 325 (1963).

(23) Y. Hara, Phys. Rev. 136, B507 (1964); L.C. Wang, Phys. Rev. 142, 1187 (1966).

(24) A.C. Hearn and E. Leader, Phys. Rev. 126, 789 (1962).

(25) B.W. Lee, Theoretical Physics, I.C.T.P. (Trieste) 1965.

(26) W. Krolikowski, Nuovo Cimento 44A, 745 (1966), ibid.42A, 435 (1966). R. Delburgo, A. Salam & J. Strathdee, Nuovo Cim. 49 593 (1967).

(27) R.C. Hwa and J. Nuyts, "On the Commutation Relations of Baryon Currents", I.A.S., Princeton, 1966.

(28) S. Fubini, Nuovo Cimento, 43A , 475 (1966).

(29) D.N. Williams, Report U.C.R.L. - 11113 (1963) (unpublished).
K. Hepp, Helv. Phys. Acta, 37, 55 (1964).
G.C. Fox, Phys. Rev., 157, 1493 (1967).

(30) M. Gell-Mann & M. Levy, Nuovo Cimento 16, 705 (1960).